UNIT 7

Equations and inequalities

Introduction

In this unit you will develop some more sophisticated mathematical problem-solving skills, and see how to apply them to some practical situations.

Formulas were discussed in Unit 2, Subsection 3.1.

In Section 1 you will learn how you can begin with an equation involving two or more variables, choose one of the variables, and turn the equation into a *formula* with that variable as its subject. This skill is often useful when you need to work with formulas, and it is useful in many other situations where you use algebra. In particular, you will see how it allows you to express the equation of a straight line in forms other than $y = mx + c$.

In Section 2 you will learn another basic algebraic technique: taking out common factors. This skill is helpful when you want to turn a given equation into a formula with a particular variable as its subject, and it is useful in other contexts too.

Section 3 introduces an important new algebraic skill that is based on many of the algebraic and graphical ideas that you have met throughout the module so far. You will learn how to deal with situations that involve *two* equations at once, and see how to apply this skill to some practical problems.

Finally, in Section 4, you will extend the work that you have done in Unit 2 on inequalities. You will learn how you can work with inequalities using similar techniques to those that you have used for equations, and how these ideas can be applied in practice.

1 Rearranging equations

1.1 Making a chosen variable the subject of an equation

In the previous unit you saw that one way to convert temperatures between the Celsius and Fahrenheit scales is to use a graph, like the one shown in Figure 1. You can use this graph to convert either from Celsius to Fahrenheit, or vice versa. For example, the dashed red lines on the graph show that 29°C is about 84°F, and the dashed blue lines show that 63°F is about 17°C.

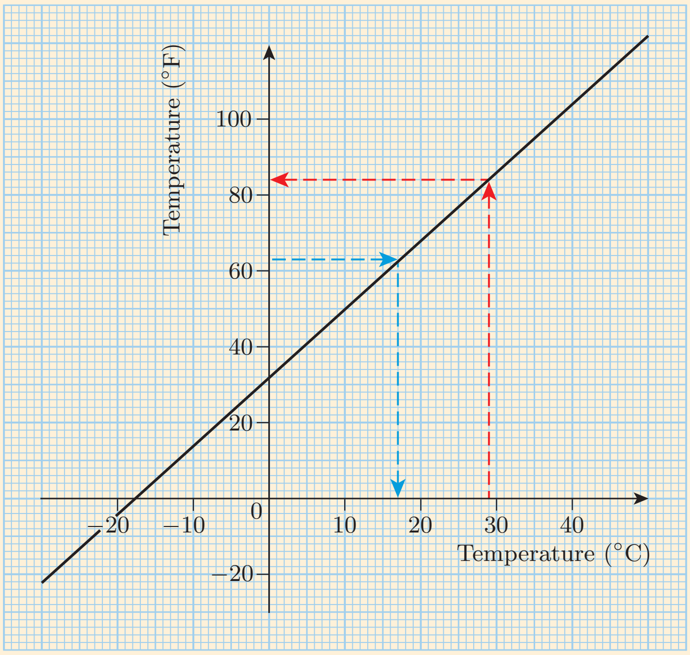


Figure 1 The Celsius-Fahrenheit conversion graph

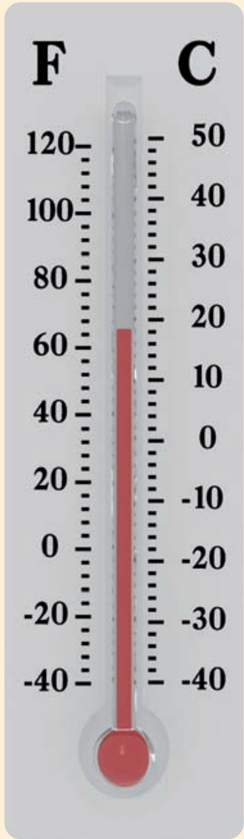


Figure 2 A thermometer with Celsius and Fahrenheit scales

If you need more accuracy than you can get from this graph, then you can use a formula instead. You have seen that if c and f denote a temperature in °C and °F, respectively, then the following formula holds:

$$f = 1.8c + 32. \tag{1}$$

For example, to find the Fahrenheit equivalent of 29°C, you substitute $c = 29$ into the formula, which gives

$$f = 1.8 \times 29 + 32 = 84.2.$$

So 29°C is the same as 84.2°F.

You can use the same formula to convert the other way, from °F to °C, but this involves solving an equation, as illustrated in the example below.

Example 1 *Solving an equation obtained from a formula*

Use formula (1) to find the Celsius equivalent of 63°F.

Solution

Substitute the value of f into the formula.

Substituting $f = 63$ into the formula gives

$$63 = 1.8c + 32.$$

Solve this equation to find c.

Subtract 32: $31 = 1.8c$

Divide by 1.8: $\dfrac{31}{1.8} = c$

Swap the sides, and do the division: $c = 17.2$ (to 1 d.p.)

So 63°F is the same as 17.2°C, to one decimal place.

Here is a similar conversion for you to try.

Activity 1 *Solving an equation obtained from a formula*

Use formula (1) to find the Celsius equivalent of 59°F.

So the formula above can be used to convert in either direction. But it is more suited to conversions from °C to °F, because that way you don't have to solve an equation. If you want to convert the other way, from °F to °C, then it would be more convenient to have a formula of the form

$$c = \text{an expression containing } f$$

– that is, a formula whose subject is c rather than f – because then you could convert from °F to °C by just substituting into the right-hand side. A formula like this is described as a formula for c in terms of f and would be especially useful if you had to make a lot of conversions from °F to °C.

This is a situation where algebra is useful! It can be used to rearrange the original formula, which has subject f, into a formula with subject c. To do this, you just 'solve' the original formula to find c in the usual way. The only difference from solving the equations in Example 1 and Activity 1 is that you don't replace f by a particular temperature – you just leave it as f. Here is what happens when you do this.

The original formula is:	$f = 1.8c + 32$
Subtract 32:	$f - 32 = 1.8c$
Divide by 1.8:	$\dfrac{f - 32}{1.8} = c$

So a formula for c is

$$c = \frac{f - 32}{1.8}. \tag{2}$$

Remember that by convention we write the subject of a formula on the left.

Activity 2 *Using the rearranged formula*

Use formula (2) to find the Celsius equivalent of 85.1°F.

Any equation that contains two or more variables gives you information about how those variables are related. It's often useful to choose one of the variables and rearrange the equation to make that variable the subject. In this rest of this subsection you'll be able to practise doing that.

As you've seen, the way to do it is to use the same method that you use to solve equations, treating the variable that you want to be the subject (which we'll call the *required subject*) as the unknown. Here's a simple example for you to try.

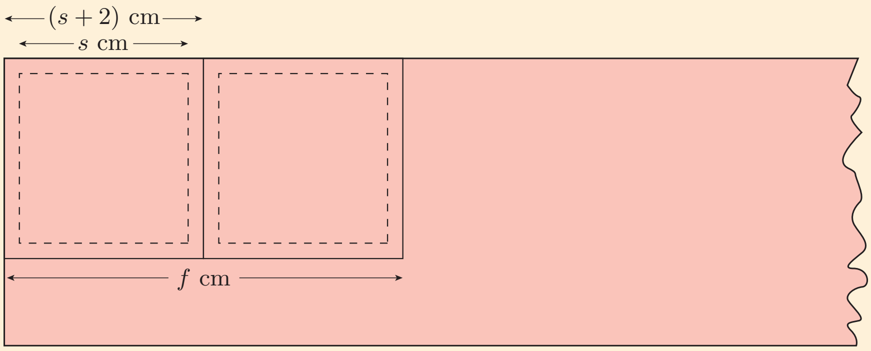
Activity 3 *Making a variable the subject of an equation*

A formula for the length f cm of fabric that you need to make a square cushion with sides of length s cm and seam allowances of 1 cm is

$$f = 2s + 4.$$

This is because you need two pieces of fabric, each of which must have length s cm plus 2 cm for the seams, as shown below.

(The formula assumes that the fabric is wide enough for each piece, as shown in the diagram, but not so wide that both pieces will fit into its width.)



Rearrange the formula above to make s the subject. (The rearranged formula allows you to work out the side length of the largest cushion that you can make with a piece of fabric of length f cm.)

The next example illustrates how to change the subject of a slightly more complicated formula. As before, the method is the same as for solving equations.

Example 2 *Making a variable the subject of another equation*

Tutorial clip

Make R the subject of the equation

$$P = \frac{Q}{R+1}.$$

Solution

The equation is: $P = \dfrac{Q}{R+1}$

First remove any fractions and brackets. To clear the fraction, multiply both sides by $R+1$. Use brackets to show that the *whole* of each side is multiplied by the *whole* of $R+1$.

Remember that it is usually best to deal with fractions before brackets, as you often have to introduce extra brackets when clearing fractions.

Multiply by $R+1$: $P(R+1) = \left(\dfrac{Q}{R+1}\right)(R+1)$

Simplify: $P(R+1) = Q$

Multiply out the brackets: $PR + P = Q$

Now there are no fractions or brackets. Next, find all the terms that contain the required subject, R. There is just one such term, PR. Keep this term on the left, and get all the other terms on the right.

Subtract P: $PR = Q - P$

Finally, divide by P, to get R by itself on the left-hand side.

Divide by P
(assuming that $P \neq 0$): $R = \dfrac{Q-P}{P}$

The symbol '$\neq$' means (and is read as) 'is not equal to'.

Notice that a 'simplify' step is included in the working in Example 2. As you saw in Unit 5 when you were solving equations, you can shorten the working a little by doing the same thing to both sides and simplifying the resulting equation all in one step. You should do this once you've had enough practice to be confident that you can do it without making mistakes.

Another thing to notice about the working in Example 2 is that an expression was divided by P. Since you cannot divide by zero, this means that the formula obtained cannot be used when $P = 0$. But it can be used for all other values of P.

The strategy used to rearrange equations in this section is summarised in the box below. It's just the strategy that you've seen for solving equations, with the wording changed slightly so that it makes sense for equations that contain more than one letter.

You can use this strategy to rearrange many of the equations that you'll need to deal with, but it won't work for all equations. Later in the unit, and later in the module, you'll see how the strategy can be adapted to deal with a wider variety of equations.

Strategy *To make a variable the subject of an equation*

Carry out a sequence of steps. In each step, do one of the following:

- do the same thing to both sides
- simplify one side or both sides
- swap the sides.

Remember that you have to do the same thing to the *whole* of each side.

Aim to do the following, in order.

1. Clear any fractions and multiply out any brackets. To clear fractions, multiply both sides by a suitable expression.
2. Add or subtract terms on both sides to get all terms containing the required subject on one side, and all the other terms on the other side. This gives an equation of the form

   $$\text{expression} \times \text{required subject} = \text{expression}.$$

3. Divide both sides by the expression that multiplies the required subject.

Remember that you can do the following things to each side.

- Add something.
- Subtract something.
- Multiply by something.
- Divide by something (provided that it is non-zero).

The 'something' can be any expression (since all expressions represent numbers).

In the next activity you're asked to rearrange a formula that is used for converting temperatures between degrees Fahrenheit and *kelvins*. As you saw in Unit 1, the kelvin is the SI unit used by scientists for measuring temperature. Temperatures measured on the Kelvin scale are in *kelvins*, not *degrees kelvin*, and the symbol used is K, not °K. The Kelvin scale uses the same increments as the Celsius scale – an increase of 1 K is the same as an increase of 1°C. The difference between the two scales is that zero on the Kelvin scale (equivalent to about −273°C) is *absolute zero*. An object with this temperature would have no heat at all.

The Kelvin scale takes its name from the scientist and engineer William Thomson, 1st Baron Kelvin (1824–1907). As a scientist Thomson made major contributions to the study of heat and thermodynamics. As an engineer he worked on the installation of the first transatlantic telegraph cable.

Activity 4 *Changing the subject of a formula*

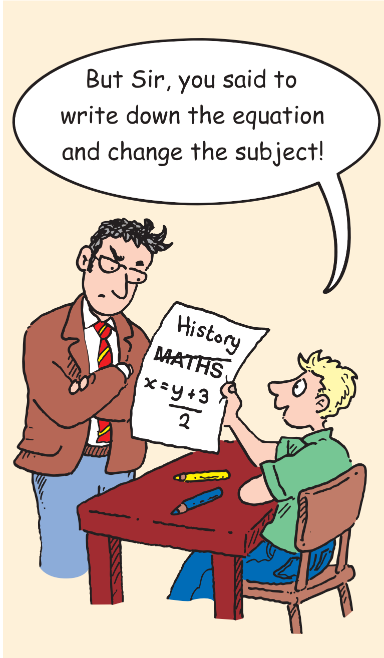
The following formula is used to convert temperatures from kelvins to degrees Fahrenheit:

$$f = 1.8(k - 273) + 32.$$

The variables f and k denote the temperatures in °F and K, respectively. (This formula is fairly accurate, but is not exact.)

(a) Make k the subject of this formula.

(b) Hence express 97.7°F in kelvins, to three significant figures.



In Unit 2 the relationship between the distance, speed and time for a journey was used in various ways. Sometimes you started with the distance and time and calculated the speed, sometimes you started with the distance and speed and calculated the time, and sometimes you started with the speed and time and calculated the distance. Now that you know how to rearrange formulas, you can remember just one formula relating the three quantities, and rearrange it into the form that you need for any particular calculation.

Activity 5 *Changing the subject of the distance–speed–time formula*

The distance d, speed s and time t for a journey are related by the formula

$$d = st.$$

(a) Make t the subject of this formula.

(b) Hence find the time that it would take to travel 96 kilometres at a speed of 80 kilometres per hour.

When you're making a variable the subject of an equation, you don't need to follow the steps of the strategy on page 128 rigidly if you can see a better way to proceed. For example, in Example 2 the equation

$$P = \frac{Q}{R+1}$$

was rearranged to make R the subject. First, both sides were multiplied by $R + 1$ to remove the fraction, which gave

$$P(R + 1) = Q.$$

At this point, instead of removing the brackets as specified by the strategy, you can proceed as follows.

Divide by P (assuming that $P \neq 0$): $\quad R + 1 = \dfrac{Q}{P}$

Subtract 1: $\quad R = \dfrac{Q}{P} - 1$

The last equation above has R as its subject, and it was obtained in three steps instead of four. It's slightly different from the equation found in Example 2, which was

$$R = \frac{Q - P}{P}.$$

You saw how to expand a fraction in Unit 5.

But naturally it's equivalent, as you can check by expanding the fraction on the right-hand side of the equation found in Example 2:

$$\frac{Q-P}{P} = \frac{Q}{P} - \frac{P}{P} = \frac{Q}{P} - 1.$$

It's best to aim to follow the strategy, but if you can see a better way to proceed at any point, then go ahead with that. You just have to remember that in each step you must do the same thing to both sides, manipulate the sides or swap the sides – you must not do anything else!

The next activity gives you more practice in rearranging equations.

Activity 6 *Rearranging equations*

(a) Make x the subject of the equation $y = \frac{x}{2} - 3$.

(b) Make r the subject of the equation $s = \frac{6}{r}$.

(c) Make Z the subject of the equation $X = \frac{Z}{Y+1}$.

(d) Make p the subject of the equation $q = \frac{2}{3}(p+2)$.

(e) Make d the subject of the equation $c = \frac{b}{2d+1}$.

1.2 Rearranging equations of straight lines

In this subsection you'll see that the skill of rearranging equations can be useful when you're dealing with the equations of straight lines.

In Unit 6 you saw that every non-vertical straight line has an equation of the form $y = mx + c$, where m is the gradient of the line and c is the y-intercept. For example, $y = 3x - 2$ is the equation of the line with gradient 3 and y-intercept -2, as shown in Figure 3.

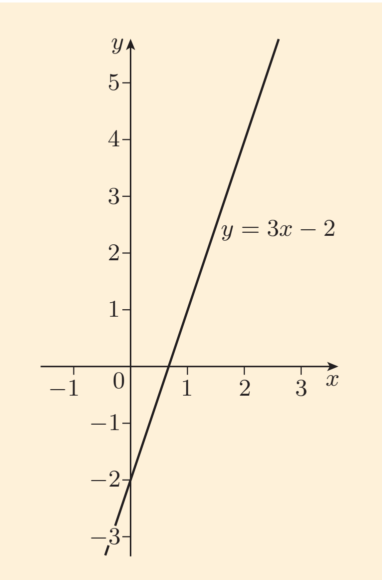


Figure 3 The line with equation $y = 3x - 2$

The line consists of all the points (x, y) that satisfy the equation $y = 3x - 2$. (Remember that a point *satisfies* a given equation in x and y if it makes the equation correct when its coordinates are substituted in.)

If you rearrange an equation in x and y, then the points that satisfy it do not change. For example, the equations $y = 3x - 2$ and $y + 2 = 3x$ are rearrangements of each other (the second is obtained by adding 2 to each side of the first), so the same points satisfy each of them. Each of these equations represents the line in Figure 3.

In the next activity you're asked to show that a particular point satisfies both of these equations.

Activity 7 *Checking that a point satisfies equations*

Show that the point $(3, 7)$ satisfies each of the following equations.

(a) $y = 3x - 2$ (b) $y + 2 = 3x$

Because rearranging an equation in x and y does not change the points that satisfy it, the following fact holds.

Any equation in x and y that can be rearranged into the form $y = mx + c$ is the equation of a straight line.

Tutorial clip

Example 3 *Rearranging the equation of a line*

Show that the equation $-5x + 3y = 4$ can be rearranged into the form $y = mx + c$. Find the gradient and y-intercept of the line that this equation represents, and draw the line.

Solution

Make y the subject of the equation, using the usual strategy.

The equation is: $-5x + 3y = 4$

Add $5x$: $3y = 5x + 4$

Divide by 3: $$y = \frac{5x + 4}{3} \qquad (3)$$

This has made y the subject, but we want the form $y = mx + c$.

Expand the fraction: $$y = \frac{5x}{3} + \frac{4}{3}$$

Simplify: $y = \frac{5}{3}x + \frac{4}{3}$

This is of the form $y = mx + c$.

Read off the gradient and y-intercept.

The gradient is $\frac{5}{3}$ and the y-intercept is $\frac{4}{3}$.

Use the two-point method to draw the line. You can use any of the rearrangements of the equation to find the two points.

The point corresponding to the y-intercept is $\left(0, \frac{4}{3}\right) \approx (0, 1.3)$. Another point can be found by substituting $x = 1$, say, into equation (3).

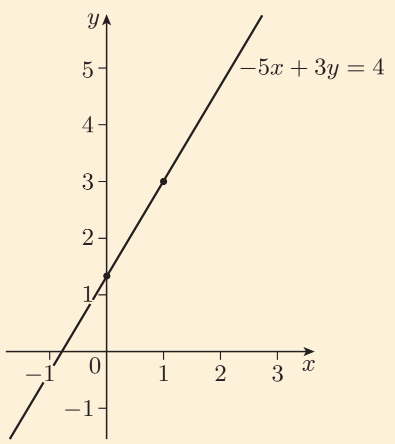
The decimal approximation $(0, 1.3)$ for $\left(0, \frac{4}{3}\right)$ is calculated to help with plotting this point.

This gives

$$y = \frac{5 \times 1 + 4}{3} = \frac{5 + 4}{3} = \frac{9}{3} = 3,$$

so another point on the line is $(1, 3)$.

The line is shown below.



In Example 3, the equation of a line was rearranged to give

$$y = \tfrac{5}{3}x + \tfrac{4}{3}.$$

The numbers in the equation were written as fractions, not as decimal approximations.

If the numbers in an equation cannot be written exactly as finite decimals, but can be written exactly as fractions, then you should write them as fractions. This is because an exact equation is better than an approximate one. If any of the fractions in an equation are top-heavy, then you should write them in top-heavy form rather than converting them to mixed numbers. For example, if an equation contains the number $1\frac{1}{3}$, then you should write it as $\frac{4}{3}$. This makes the equation clearer to read.

If the numbers in an equation can be written exactly as either fractions or decimals, then you can use either. So, for example,

$$y = \tfrac{1}{2}x - \tfrac{7}{5} \quad \text{and} \quad y = 0.5x - 1.4$$

are both acceptable forms of the same equation.

As you have seen in earlier units, in mathematics it is usually best to work with exact numbers, unless there is a good reason to use approximations. In Example 3, a decimal approximation for a coordinate of a point was calculated to help with plotting, but exact numbers were used everywhere else.

Activity 8 *Rearranging the equation of a line*

Show that the equation $3x + 2y = 6$ can be rearranged into the form $y = mx + c$. Find the gradient and y-intercept of the line that this equation represents, and draw the line.

The equations in Example 3 and Activity 8 are both of the form $ax + by = d$, where a, b and d are numbers (positive or negative). Any equation of this form with $b \neq 0$ can be rearranged into the form $y = mx + c$. If b *is* zero, then provided that $a \neq 0$, the equation can be rearranged into the form $x = c$ for some number c. So we have the following fact.

> An equation of the form $ax + by = d$, where a, b and d are numbers with a and b not both zero, is the equation of a straight line.

If you're sure that an equation in x and y is the equation of a straight line, then you don't need to rearrange it into the form $y = mx + c$ before you draw the line. You can use the equation in its original form to find two points on the line, and hence draw it.

Example 4 *Drawing a line from its equation*

Draw the line with equation $-x + 4y = 2$.

Solution

Find two points on the line. It's often easiest to find the point with $x = 0$ and the point with $y = 0$.

Substituting $x = 0$ into the equation gives

$$4y = 2.$$

Divide by 4: $y = \frac{1}{2}$

So a point on the line is $(0, \frac{1}{2})$.

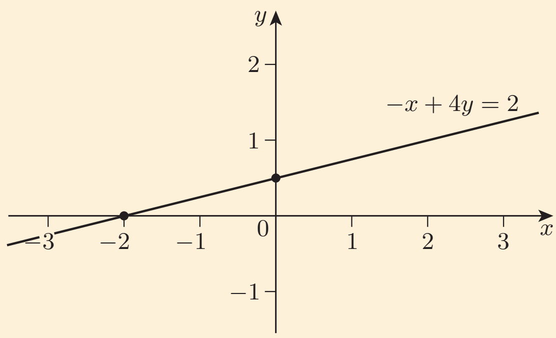
Substituting $y = 0$ into the equation gives

$$-x = 2.$$

Multiply by -1: $x = -2$.

So a point on the line is $(-2, 0)$.

The line is shown below.



Here's a similar example for you to try.

Activity 9 *Drawing a line from its equation*

Draw the line with equation $4x - 3y = -6$.

In this section you have learned a method for rearranging equations. It was explained that although this method can be used for many equations, an improved method is needed to allow you to deal with a wider variety of equations. The next section introduces a new technique that is needed for this method.

2 Common factors

The new technique that you need for rearranging equations is called *taking out common factors*, and is the reverse of multiplying out brackets. For example, if you have the expression $x^2 + x$, then you can use the technique to write it as $x(x + 1)$.

The technique of taking out common factors is useful not only for rearranging equations, but in many other situations where algebra is used. In this section you'll learn about this technique and you'll see how you can use the skill of rearranging equations to find the answers to some practical problems.

2.1 Finding common factors

The way to reverse the process of multiplying out brackets is to consider the *factors* of the terms in the expression. A factor of a term is similar to a factor of an integer, which was discussed in Unit 3.

Remember that a positive integer that divides a given integer exactly is called a *factor* of that integer. For example, 4 and 5 are both factors of 20 because

$$20 = 4 \times 5.$$

A similar idea applies to terms. If a term can be written in the form

$$\text{something} \times \text{something},$$

then both 'somethings' are **factors** of the term.

For example, consider the term a^2b. We can write

$$a^2b = a \times ab,$$

so a and ab are factors of a^2b. Similarly,

$$a^2b = a^2 \times b \quad \text{and} \quad a^2b = 1 \times a^2b,$$

so a^2, b, 1 and a^2b are also factors of a^2b.

Example 5 *Checking a factor*

Show that $3xy$ is a factor of $3xy^3$, by writing $3xy^3$ in the form

$$3xy \times \text{something}.$$

Solution

$$3xy^3 = 3xy \times y^2$$

Activity 10 *Checking factors*

(a) Write pqr in the form $q \times$ something.

(b) Write A^7 in the form $A^4 \times$ something.

(c) Write $4f^3$ in the form $2f \times$ something.

(d) Write p^3q^5 in the form $p^2q^2 \times$ something.

(e) Write $6x^5y^8$ in the form $2x^2y \times$ something.

Remember also from Unit 3 that if a positive integer is a factor of *each* of several integers, then it is a *common factor* of those integers. For example, 2 is a common factor of 8, 12 and 20, because

$$8 = 2 \times 4, \quad 12 = 2 \times 6 \quad \text{and} \quad 20 = 2 \times 10.$$

Again, a similar idea applies to terms. If something is a factor of *each* of several terms, then it is a **common factor** of those terms. For example, a is a common factor of

$$a^2b \quad \text{and} \quad abc,$$

because

$$a^2b = a \times ab \quad \text{and} \quad abc = a \times bc.$$

Example 6 *Checking a common factor*

Show that pq is a common factor of pq^2, $3p^2q^2$ and pq.

Solution

Write each term in the form $pq \times$ something.

$$pq^2 = pq \times q, \quad 3p^2q^2 = pq \times 3pq \quad \text{and} \quad pq = pq \times 1.$$

So pq is a common factor of the three terms.

Activity 11 *Checking common factors*

(a) Show that z is a common factor of $2z$ and z^2.

(b) Show that p^2 is a common factor of p^2q^2 and p^2.

(c) Show that $2AB$ is a common factor of $2A^2B^2$, $4A^2B$ and $8AB$.

In Unit 3 you saw that a collection of integers can have several common factors, and the largest of them is called the *highest* common factor. For example, the common factors of 8, 12 and 20 are 1, 2 and 4, and the highest common factor is 4.

Again, this idea also applies to terms: you can find the highest common factor of several terms. For example, consider again the terms

$$a^2b \quad \text{and} \quad abc.$$

You have seen that one common factor of these terms is a, because

$$a^2b = a \times ab \quad \text{and} \quad abc = a \times bc.$$

Another common factor is ab, because

$$a^2b = ab \times a \quad \text{and} \quad abc = ab \times c.$$

The common factor ab is a *higher* common factor than a, as it is a multiplied by another factor. In fact, ab is the **highest common factor** of the two terms, since you cannot multiply ab by any other letters or integers (except 1) and still get a common factor of the two terms.

The next example shows you how to find the highest common factor of two terms.

Tutorial clip

Example 7 *Finding a highest common factor*

Find the highest common factor of the terms

$$6ab^7c^2 \quad \text{and} \quad 9a^2b^5,$$

and write each term in the form

highest common factor $\times$ something.

Solution

First, consider the coefficients. The largest integer that divides both 6 and 9 exactly is 3.

3 is the highest common factor of 6 and 9.

Next, consider the powers of a. The largest power of a that divides both a and a^2 exactly is a.

Then consider the powers of b. The largest power of b that divides both b^7 and b^5 exactly is b^5.

Finally, consider the powers of c. The second term doesn't contain c at all.

So, the highest common factor of the two terms is

$$3ab^5.$$

The terms can be written as

$$6ab^7c^2 = 3ab^5 \times 2b^2c^2 \quad \text{and} \quad 9a^2b^5 = 3ab^5 \times 3a.$$

Activity 12 *Finding highest common factors*

In each of the following parts, find the highest common factor of the terms and write each term in the form

highest common factor × something.

(a) $2ab^2$ and $4ab$ (b) $3xy$ and $6y$

(c) $4p^3$, $9p^2$ and $2p^5$ (d) $10r$ and $15s$

2.2 Taking out common factors

Consider the expression

$$c^2d + cef.$$

The two terms in this expression, c^2d and cef, have c as a common factor. So the expression can be written as

$$c \times cd + c \times ef.$$

From your work on multiplying out brackets, you know that this is the same as

$$c(cd + ef).$$

You can now see how to reverse the process of multiplying out brackets. First, you find a common factor of the terms, and write each term in the form

common factor × something.

Then you write the common factor at the front of a pair of brackets, and inside the brackets you write what is left of each term (the 'something'). This is called taking out a common factor, or **factorising** the expression.

Tutorial clip

Example 8 *Taking out a common factor*

Factorise the expression $3rs^3 + rs$.

Solution

A common factor of the terms is rs.

$$\begin{aligned} 3rs^3 + rs &= rs \times 3s^2 + rs \times 1 \\ &= rs(3s^2 + 1) \end{aligned}$$

You can take out any common factor of the terms in an expression, but it is usually best to take out the highest common factor. Do this in the next activity. Remember that you can always check a factorisation by multiplying out the brackets and checking that the expression that you get is the same as the one you factorised.

Activity 13 *Taking out common factors*

Factorise the following expressions.

(a) $ab + a^2$ (b) $x^3y + yz$ (c) $2w^2 + w^3$ (d) $2z + 6z^4$

Expressions containing minus signs can be factorised in the same way.

Example 9 *Factorising an expression containing minus signs*

Tutorial clip

Factorise the expression $3m^3 - 6m^2 + 3m^4$.

Solution

The highest common factor of the terms is $3m^2$.

$$\begin{aligned} 3m^3 - 6m^2 + 3m^4 &= 3m^2 \times m - 3m^2 \times 2 + 3m^2 \times m^2 \\ &= 3m^2(m - 2 + m^2) \end{aligned}$$

Activity 14 *Factorising expressions containing minus signs*

Factorise the following expressions.

(a) $2ab + 2b - 6b^2$ (b) $A^5 - A^4$

As you get used to taking out common factors, you'll probably find that you can skip the step of writing each term in the form

common factor $\times$ something,

and just follow the shorter strategy below.

Strategy *To take out a common factor from an expression*

1. Find a common factor of the terms (normally the highest common factor).
2. Write the common factor in front of a pair of brackets.
3. Write what's left of each term inside the brackets.

Tutorial clip

Example 10 *Factorising efficiently*

Factorise the expression $-8X^2 + 2X + 2XY$.

Solution

The highest common factor is $2X$.

$$-8X^2 + 2X + 2XY = 2X(-4X + 1 + Y)$$

When you use the shorter strategy above, remember that if the common factor that you're taking out is the same as one of the terms, then 'what's left' of the term is 1. In Example 10, taking the factor $2X$ out of the term $2X$ left 1.

Try the shorter strategy in the following activity.

Activity 15 *Factorising efficiently*

Factorise the following expressions.

(a) $ab - 9bc$ (b) $x^2 - x^5 + 2x^3$ (c) $-2rs + 4r^2s^2$ (d) $x\sqrt{y} - \sqrt{y}$

Once you have factorised an expression, you should check that you have taken out the *highest* common factor. To do this, check whether the terms inside the brackets have a common factor. If they do, take it out as well.

For example, suppose that you have carried out the following factorisation:

$$de^2 - d^2e^2 + de^3 = de(e - de + e^2).$$

The terms inside the brackets have the common factor e, so you did not take out the highest common factor. Your working would continue as follows.

$$\begin{aligned} &= de \times e(1 - d + e) \\ &= de^2(1 - d + e). \end{aligned}$$

Now the terms inside the brackets have nothing in common, so de^2 is the highest common factor that can be taken out.

Many expressions cannot be factorised at all. For example, the terms in the expression

$$2de + 3ef + 4fd$$

have no common factor that can be usefully taken out.

Activity 16 *Taking out more common factors*

Factorise the following expressions, where possible.

(a) $12u + 6u^3 - 9u^2$ (b) $5r^2 - 10$

(c) $3fg - 2gh + 6fh$ (d) $-8ABC - 4AB^2 + 2AB$

If the coefficients of the terms of an expression are not integers, then you can often still factorise the expression.

Example 11 *Working with non-integer coefficients*

Factorise the following expressions.

(a) $0.2a - 0.8a^2$ (b) $\frac{1}{2} - \frac{3}{2}q$

Solution

(a) $0.2a - 0.8a^2 = 0.2a(1 - 4a)$

(b) $\frac{1}{2} - \frac{3}{2}q = \frac{1}{2}(1 - 3q)$

Activity 17 *Working with non-integer coefficients*

Factorise the following expressions.

(a) $0.3m^2 - 0.6m + 0.9$ (b) $\frac{1}{2}x - \frac{1}{2}x^2$

Earlier in the module you saw how to multiply out brackets with a minus sign in front. You just change the sign of each term in the brackets. For example,

$$-(a + 2b - 2c - d) = -a - 2b + 2c + d. \qquad (4)$$

Sometimes it's useful to carry out the reverse of this process: to start with an expression without brackets and rewrite it so that it has a minus sign in front of brackets.

For example, you could start with the expression on the right-hand side of equation (4) and rewrite it as the expression on the left-hand side. As you can see, to do this you just have to change the sign of each term in the brackets. In general, remember the following.

To take a minus sign outside brackets

Change the sign of each term in the brackets.

Taking a minus sign outside brackets is the same as taking out a factor of -1.

For example,

$$-1 - x - x^2 + x^3 = -(1 + x + x^2 - x^3).$$

If you want to factorise an expression, and all or most of the terms have minus signs, then it's usually best to take out a minus sign as well as any other common factors. In the next example first a minus sign is taken out, and then a common factor. As you get more used to taking out common factors, you should find that you can do both these things in one step.

Example 12 *Taking out a minus sign*

Factorise the expression $-a - ab + a^2 - a^3$.

Solution

First take out a minus sign, then take out the common factor a.

$$\begin{aligned} -a - ab + a^2 - a^3 &= -(a + ab - a^2 + a^3) \\ &= -a(1 + b - a + a^2) \end{aligned}$$

Activity 18 *Taking out minus signs*

For each of the following expressions, take out a minus sign and factorise the expression if possible.

(a) $-2u^2 - 2u^3 - 4u^4$ (b) $-1 - a + a^2$ (c) $pq - p^2q - q^2p - p^2q^2$

When you're factorising expressions, remember that you can always check your answer. Just multiply out the brackets again and check that you get the original expression. A check is particularly useful when you've taken out a minus sign.

2.3 An improved strategy for rearranging equations

Now that you've learned how to take out common factors, you're ready to learn how to rearrange a wider variety of equations. Here is an example of an equation for which taking out common factors is useful:

$$5x = 2z + xy. \tag{5}$$

Suppose that you want to make x the subject of this equation. You can't do this by dividing each side by 5, because although that would give x by itself on the left-hand side, there would still be another occurrence of x on the right-hand side. Remember from Unit 2 what it means for a variable to be the subject of an equation – the definition can be expressed as follows.

The reason for this definition is that an equation with a subject should be of a form that allows you to substitute in for the variables other than the subject, and hence work out the value of the subject, without having to solve an equation.

A subject of an equation is a variable that appears by itself on one side, and not at all on the other side.

Let's think about making x the subject of equation (5) by applying the strategy that you saw in Subsection 1.1, on page 128. The equation has no fractions or brackets, so the first thing to do is to make sure that all the terms containing x are on one side, and all the other terms are on the other side. You can do this by subtracting the term xy from each side, which gives

$$5x - xy = 2z.$$

In this equation, all the terms containing x are on the left, and all the other terms are on the right. According to the strategy, the next step is to divide both sides by the expression that multiplies x. But there isn't a single expression multiplying x, because x appears *twice* on the left-hand side.

You can solve this problem by taking out x as a common factor on the left-hand side. This gives

$$x(5 - y) = 2z.$$

Now there is an expression multiplying x, namely $5 - y$, and you can divide both sides by this expression to give

$$x = \frac{2z}{5 - y},$$

which is an equation with x as its subject.

(Because this equation was obtained by dividing by $5 - y$, it's not valid when $5 - y = 0$; that is, when $y = 5$. But it can be used for all other values of y.)

Here is the strategy from Subsection 1.1, amended to take account of the fact that you might need an extra step in which you take out a common factor.

Strategy *To make a variable the subject of an equation*

Carry out a sequence of steps. In each step, do one of the following:

- do the same thing to both sides
- simplify one side or both sides
- swap the sides.

Aim to do the following, in order.

1. Clear any fractions and multiply out any brackets. To clear fractions, multiply both sides by a suitable expression.
2. Add or subtract terms on both sides to get all terms containing the required subject on one side, and all the other terms on the other side.
3. If more than one term contains the required subject, then take it out as a common factor. This gives an equation of the form

$$\text{expression} \times \text{required subject} = \text{expression}.$$

4. Divide both sides by the expression that multiplies the required subject.

When you use this improved strategy, you need to be especially careful when carrying out step 2. In this step it's essential to make sure that you get *all* the terms containing the required subject on one side, and *all* the terms not containing the required subject on the other side. Check carefully that you have done this before moving on to step 3. Here's an example.

Example 13 *Making a variable the subject of an equation*

Tutorial clip

Make c the subject of the equation $2c - a = bc + 1$.

Solution

Use the strategy above.

The equation is: $\quad 2c - a = bc + 1$

There are no fractions or brackets, so step 1 isn't needed. Move on to step 2. The term $2c$ on the left and the term bc on the right both contain the required subject, c, so first get both of these terms on the left.

Subtract bc: $\quad 2c - a - bc = 1$

The term $-a$ on the left doesn't contain the required subject, c, so get it on the right.

Add a: $\quad 2c - bc = a + 1$

Check that all the terms that contain the required subject, c, are on the left, and all the terms that don't contain the required subject are on the right. They are, so step 2 has been completed. Move on to step 3: take out the required subject as a common factor.

Take out c as a common factor: $\quad c(2-b)=a+1$

Finally, do step 4: divide by the expression that multiplies the required subject.

Divide by $2-b$ (assuming that $b \neq 2$): $\quad c=\dfrac{a+1}{2-b}$

As a check, confirm that the subject, c, appears by itself on the left-hand side, and *not at all on the right-hand side.*

The next activity will give you more practice in rearranging equations. You will need to use many of the algebraic skills that you have learned so far. Follow the strategy carefully!

Activity 19 *Making a variable the subject of an equation*

(a) Make h the subject of the equation $gh=g+h$.

(b) Make y the subject of the equation $x=\dfrac{x-y}{y}$.

(c) Make s the subject of the equation $r=\dfrac{s}{r}+2s$.

(d) Make p the subject of the equation $2p+q=r(p+q)$.

In the next two activities you're asked to use the skills that you have learned in this section, together with skills that you learned earlier in the module, to find the answers to some practical problems.

Activity 20 *How many partygoers?*

A club is organising a party. It costs £100 to hire the venue, and £10 per person for refreshments. Five special guests will not be asked to contribute to the cost, but each other partygoer will be asked to pay an equal share of the total cost of the party.

(a) Find a formula for p, where £p is the amount paid by each paying partygoer if there are n partygoers altogether (including the special guests).

(b) Make n the subject of the formula found in part (a).

(c) Use the formula found in part (b) to find the number of partygoers who must attend if the cost for each paying partygoer is to be £12.50.

Activity 21 *How many game sales?*

A company proposes to manufacture a particular type of electronic game. There will be a fixed cost of £f to set up the production line. Thereafter

each game will cost another £c to manufacture, and will be sold for £s. Let the number of games manufactured be n.

(a) Find an expression for the total cost in £ of manufacturing the n games.

(b) Find an expression for the total selling price in £ of the n games.

(c) Use your answers to parts (a) and (b) to find a formula for p, where £p is the profit made from selling n games.

(d) Make n the subject of the formula found in part (c).

(e) Use the formula from part (d) to find the number of games that must be made and sold to make a profit of £20 000, if the fixed cost is £10 000, the additional manufacturing cost of each game is £15 and the selling price of each game is £40.

In this section you have learned how to take out common factors, and you have used this technique to help you to rearrange equations. In the next section you'll return to the topic of *solving* equations – that is, finding the values of the unknowns in them.

3 Simultaneous linear equations

In Unit 5 you saw some situations where it was helpful to solve linear equations in one unknown, and you learned how to solve equations of this type. You'll now extend this work by learning how to solve *pairs* of linear equations, in *two* unknowns, where both equations hold at the same time. These are called **simultaneous linear equations**. You will learn algebraic and graphical methods of solving equations of this type. The algebraic methods involve the skills for rearranging equations that you have met in this unit.

We begin with an example of a situation that leads to simultaneous linear equations. You'll see some more examples of such situations later in the section.

3.1 A problem involving simultaneous equations

Suppose that a group of friends is planning to travel from one town to another town some distance away, as part of a holiday trip. One of the group, Fred, is a keen cyclist, and wants to make the trip on his bicycle. The rest of the group will travel in a car. Fred plans to set off at 10 am, and expects to be able to travel at a speed of about 30 km/h, whereas the rest of the group plan to set off an hour and a half later, at 11.30 am, and expect to be able to travel at a speed of about 90 km/h. Fred and the car will take the same route. The whole group want to meet for a picnic break, so they want to work out where and when the car will catch up with Fred.

You can tackle this problem by setting up a simple model for the progress of Fred and the car. Although in practice both Fred and the car will have to slow down and speed up according to the traffic conditions, for the purpose of the model let's assume that they travel at *constant speeds* of 30 km/h and 90 km/h, respectively.

You saw in Unit 2 that if the speed of a vehicle is constant, then its distance, speed and time are related by the formula

$$\text{distance} = \text{speed} \times \text{time}.$$

So if we use t to represent the time in hours since Fred set off, and d to represent the distance in kilometres from the starting point at that time, then d and t are related by the formula

$$d = 30t.$$

So you can draw a graph of Fred's progress by putting the time t and the distance d on the horizontal and vertical axes, respectively, and plotting the line with equation $d = 30t$. This is the straight line through the origin with gradient 30, as shown in Figure 4.

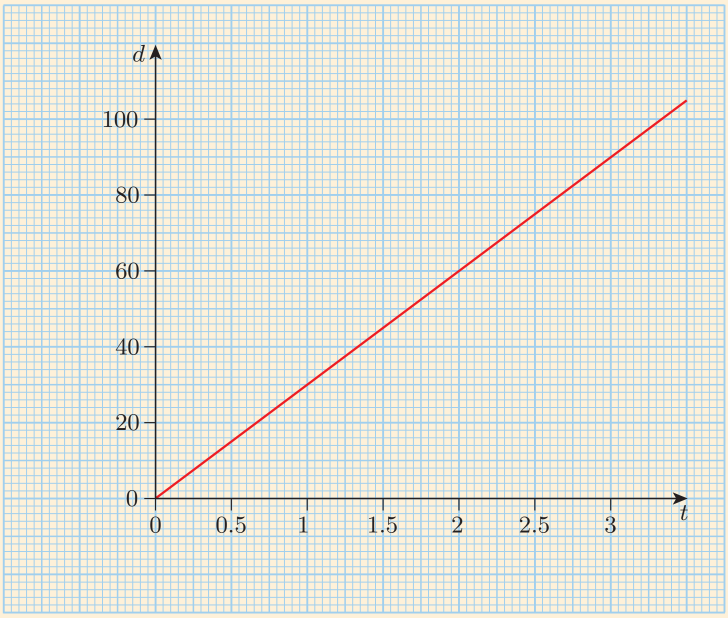


Figure 4 A graph showing the progress of the bicycle

A graph of distance against time, like that in Figure 4, is known as a distance–time graph. From the graph in Figure 4 you can read off how far Fred has travelled after any given time – for example, after 2 hours he has travelled about 60 km.

One way to try to find when and where the car will catch up with the bicycle is to draw the graph representing the progress of the car on the *same axes* as the graph of the bicycle. So we need to add the car's graph to the axes in Figure 4. Now if the variable t represented the time *since the car set off*, then the car's graph would be the straight line through the origin with gradient 90, but in fact t represents the time since *Fred* set off, which is $1\frac{1}{2}$ hours earlier, so the car's graph is shifted along the time axis by $1\frac{1}{2}$ hours (to the right).

To see why a bit more clearly, let's use the formula

$$\text{distance} = \text{speed} \times \text{time}$$

again. The car sets off 1.5 hours later than Fred, and at that time it has been travelling for 0 hours. Similarly, 2 hours after Fred set off, the car has been travelling for 0.5 hours; 2.5 hours after Fred set off, the car has been travelling for 1 hour; and so on. In general, at time t the car has been

travelling for $t - 1.5$ hours. So, since the car travels at 90 km/h, its equation is

$$d = 90(t - 1.5),$$

which is the same as

$$d = 90t - 135.$$

This is the equation of the straight line through the point $(1.5, 0)$ with gradient 90.

Figure 5 shows Fred's graph and the car's graph on the same axes.

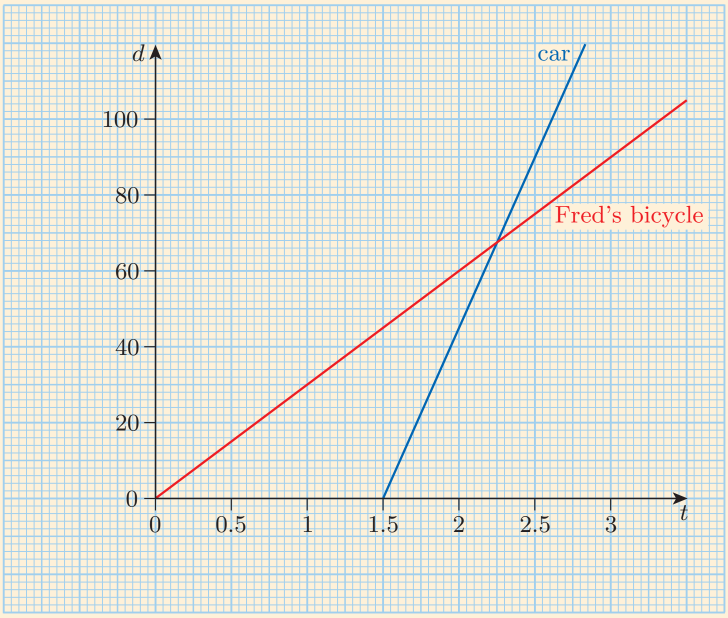


Figure 5 Graphs showing the progress of the bicycle and the car

From the distance–time graph in Figure 5 you can read off how far both Fred's bicycle and the car have travelled after any given time since Fred set off. For example, after 2 hours Fred and the car have travelled about 60 km and 45 km, respectively, from the starting point.

Activity 22 *Using the graphs for the bicycle and the car*

Use Figure 5 to answer the following questions.

(a) Roughly how many hours and minutes after Fred sets off will the car catch up with him? What time will this be?

(b) At roughly what distance from the starting point will this happen?

In Activity 22 you used a graph to find the approximate time and distance at which the car will catch up with the bicycle. In doing this you assumed that the bicycle and the car were travelling at constant speeds, which is unlikely to happen in practice. However, an approximate answer is perfectly adequate for this situation – you just need to work out roughly when the car will catch up with the bicycle and don't need a time to the nearest second or a distance to the nearest metre!

There is evidence that the Babylonians as early as 1750 BC dealt with problems that we would nowadays consider as simultaneous equations.

An alternative way to find the time and distance when the car catches up is to use algebra to solve the equations directly, without using a graph. We have arrived at one of the crucial ideas in this unit: that of simultaneous linear equations and the techniques available for solving them algebraically. One advantage of using algebra is that you can find more accurate values for the solutions than you can from a graph. This is important in some situations, as you will see in some examples later in the unit.

Every point on the bicycle's line represents a time t and a distance d such that $d = 30t$, and every point on the car's line represents a time t and a distance d such that $d = 90t - 135$.

When the car catches up with the bicycle, they are the same distance d from the starting point at the same time t; so the equations describing the progress of the bicycle and the car are both correct for those values of t and d. That is, the values of t and d that we're looking for are the values such that $d = 30t$ *and* $d = 90t - 135$. If we can find such values, that simultaneously **satisfy** the two equations, then we have solved our problem. Mathematically, the problem is usually expressed as follows:

Solve the simultaneous equations

$$d = 30t, \tag{6}$$

$$d = 90t - 135. \tag{7}$$

The values of t and d that simultaneously satisfy both equations are together called the **solution** of the simultaneous equations, and the process of finding the solution is called **solving** the simultaneous equations. Equations (6) and (7) are called *linear* equations because each term in them is either a constant term or a number times one of the unknowns, d and t.

The problem has now been expressed in terms of two linear equations and two unknowns, d and t. Generally speaking, if there are the same number of linear equations as unknowns, then this gives you enough information to solve the problem (though there are exceptions, as you will see).

You have already seen in Unit 5 how to find the solution of *one* linear equation involving *one* unknown. The next step may be rather surprising if you haven't encountered simultaneous equations before; but it's possible to use the information from equations (6) and (7) to obtain a linear equation involving only one unknown.

The crucial thing to notice is that the left-hand sides of equations (6) and (7) are equal (they are both simply d). So the right-hand sides, being each equal to d, must be equal to each other! This gives us the equation

$$30t = 90t - 135. \tag{8}$$

Since d does not appear in this equation, we say that we have **eliminated** the unknown d from equations (6) and (7).

Activity 23 *Finding the time and distance algebraically*

(a) Solve equation (8); that is, find the value of t.

(b) Substitute the value of t into equation (6), and hence find the value of d.

(c) Hence write down the time when the car will catch up with the bicycle, and the distance from the starting point at this time.

(d) Check your answer, by checking that the LHS and RHS of equation (6) are equal when you substitute in the values of t and d that you have found, and that the LHS and RHS of equation (7) are also equal when you substitute in these values of t and d.

To reinforce your understanding of these processes (reducing two equations to one, and checking that your solution is correct), try the next activity.

Activity 24 *Solving simultaneous equations*

Solve the following pair of simultaneous equations:

$$y = 2x - 3,$$
$$y = 5x + 9.$$

So if you have to solve two simultaneous equations, and each of the equations is a formula for one of the unknowns in terms of the other (with the *same* unknown being the subject of the formula in each case), then solving them is likely to be reasonably straightforward, as the pair can be reduced to a single equation.

Or is it so straightforward? Consider again the journey of the group of friends. Suppose that instead of travelling by bicycle, Fred travels in another car. He sets off at the same time as before, 10 am, but travels at a steady 90 km/h. Then the lines representing Fred's progress and that of the original car are *parallel*, as shown in Figure 6.

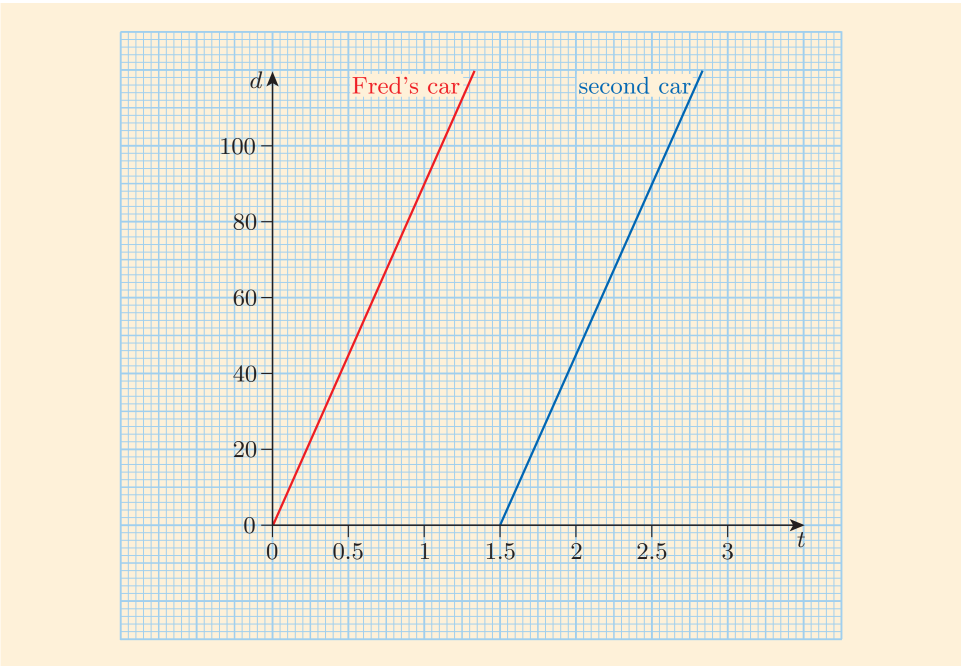


Figure 6 Graphs showing the progress of the two cars

As a matter of common sense, the second car will never catch up with Fred now (at least, not until he stops!); but there are still two simultaneous equations in two unknowns, so how does the mathematics sort this out?

The two equations are now

$$d = 90t, \tag{9}$$
$$d = 90t - 135. \tag{10}$$

Again, the two right-hand sides are equal, which gives the equation

$$90t = 90t - 135.$$

Subtracting $90t$ from each side of this equation gives the result

$$0 = -135.$$

This statement is mathematical nonsense! It says that two obviously unequal numbers are equal. Since this nonsense came from the assumption that there *is* a solution to equations (9) and (10), the conclusion that has to be drawn is that this pair of equations *has no solution*; that is, there is *no* value of t that satisfies both equations. In other words, equations (9) and (10) can't both be true at once.

So, there's a restriction on when two simultaneous equations can be solved.

How to tell whether simultaneous equations have a solution

Suppose that two simultaneous equations (in the unknowns x and y) are written in the form

$$y = ax + b,$$
$$y = cx + d.$$

These equations are the equations of straight lines with gradients a and c, respectively.

- If the constants a and c are *not equal*, then the lines representing the equations are not parallel, so the equations have one solution.
- If the constants a and c are *equal*, then the lines representing the equations are parallel, so the equations do *not* have a solution. (There is an exception to this: if the constants b and d are also equal, then the two equations are the same, so there are infinitely many solutions.)

Activity 25 *Recognising how many solutions there are*

State how many solutions each of the following pairs of simultaneous equations has. You are not asked to *find* any solutions.

Remember that each single solution of a pair of simultaneous equations is a *pair* of values – one value for each of the two unknowns.

(a) $y = -2x + 5$
$y = -2x - 1$

(b) $y = -2x + 5$
$y = 5 - 2x$

(c) $y = -2x + 5$
$y = 3x - 1$

(You might like to use Graphplotter to check your answers – you can use the 'Two graphs' tab to plot two lines on the same axes.)

In the rest of this section you'll learn how to solve pairs of simultaneous linear equations, even when they don't have the simple form of the pairs that you have seen so far. There are two methods that you can use to do this – *substitution* and *elimination*. In both methods you first find the value of one of the unknowns, because once you have such a value you can substitute it into one of the equations to find the value of the other unknown. In each method you use a different technique to find the value of the first unknown.

3.2 Substitution method for simultaneous equations

This subsection is about the substitution method for solving simultaneous linear equations. Here's an example to illustrate the idea. In this example *just one* of the equations is a formula for one of the unknowns in terms of the other.

Tutorial clip

Example 14 *Solving simultaneous equations by substitution*

Solve the simultaneous equations

$$9A - 2B = 12, \qquad (11)$$

$$B = 5A - 14. \qquad (12)$$

Solution

The second equation tells you that B is equal to $5A - 14$. So in the first equation, replace B by $5A - 14$.

Using equation (12) to substitute for B in equation (11) gives

$$9A - 2(5A - 14) = 12.$$

We now solve this equation.

$$\begin{aligned} 9A - 2(5A - 14) &= 12 \\ 9A - 10A + 28 &= 12 \\ -A + 28 &= 12 \\ -A &= -16 \\ A &= 16 \end{aligned}$$

Now substitute the value of A into either of the two equations to find the value of B. Equation (12) gives the easier calculation.

Substituting $A = 16$ into equation (12) gives

$$B = 5 \times 16 - 14 = 80 - 14 = 66.$$

So the solution is

$$A = 16,\ \ B = 66.$$

(Check: Substituting $A = 16$, $B = 66$ into equation (11) gives

$$\text{LHS} = 9A - 2B = 9 \times 16 - 2 \times 66 = 144 - 132 = 12 = \text{RHS.})$$

There is no need to check equation (12) by substituting into it because this calculation has already been done when finding the value of B.

There is a similar activity for you to try overleaf. When you write out your answer to this activity, and whenever you solve a pair of simultaneous equations, it's helpful to begin by writing down the two equations and labelling them with numbers in brackets, as you've seen in Example 14 and earlier. Then you can use these labels to refer to the equations, to help make your working clear and concise. It's often useful to label later equations in your working too. All the equations in this unit are numbered consecutively, but when you are assigning your own labels it's fine to use the numbers (1), (2), (3), and so on, each time.

Another thing to notice about Example 14 is that the solution has been written concisely by omitting the instructions 'Multiply out the brackets', 'Subtract 28', and so on, for solving the single linear equation in the unknown A. Once you feel confident about solving linear equations, feel free to do this too!

Activity 26 *Solving simultaneous equations by substitution*

Solve the simultaneous equations

$$\begin{aligned} x &= 2y - 7, \\ 3x - y &= -6. \end{aligned}$$

You now have the means at your disposal to deal with *any* pair of simultaneous linear equations in two unknowns. Even if neither of the equations is a formula for one of the unknowns in terms of the other, you can just take one of the equations and rearrange it to give such a formula! Then you can proceed as in Example 14 and Activity 26. Here's an example.

Tutorial clip

Example 15 *Solving more simultaneous equations by substitution*

Solve the simultaneous equations

$$2x + 4y = 8, \tag{13}$$
$$-3x + 5y = -1 \tag{14}$$

by the substitution method.

Solution

Choose one of the equations and rearrange it to make one of the unknowns the subject.

We make x the subject of equation (13).

$$\begin{aligned} 2x + 4y &= 8 \\ 2x &= 8 - 4y \\ x &= \frac{8 - 4y}{2} \\ x &= \frac{8}{2} - \frac{4y}{2} \end{aligned}$$
$$x = 4 - 2y \tag{15}$$

Use this formula to substitute for x in the *other* equation.

Substituting for x in equation (14) gives

$$-3(4 - 2y) + 5y = -1.$$

We now solve this equation.

$$\begin{aligned} -3(4 - 2y) + 5y &= -1 \\ -12 + 6y + 5y &= -1 \\ -12 + 11y &= -1 \\ 11y &= 11 \\ y &= 1 \end{aligned}$$

Substitute this value for y into one of the equations in x and y to find the value of x. Equation (15) gives the easiest calculation.

Substituting $y = 1$ into equation (15) gives

$$x = 4 - 2 \times 1 = 4 - 2 = 2.$$

So the solution is

$$x = 2, \ y = 1.$$

(Check: Substituting $x = 2$, $y = 1$ into equation (13) gives

$$\text{LHS} = 2 \times 2 + 4 \times 1 = 8 = \text{RHS}.$$

Substituting the same values into equation (14) gives

$$\text{LHS} = -3 \times 2 + 5 \times 1 = -1 = \text{RHS}.)$$

One thing worth noticing about the simultaneous equations in Example 15 is that all the terms in equation (13) have a common factor of 2. So both sides of this equation can be divided by 2 to give an equivalent but slightly simpler equation, as follows:

The equation is:	$2x + 4y = 8$
Divide by 2:	$\dfrac{2x + 4y}{2} = \dfrac{8}{2}$
Expand the fraction:	$\dfrac{2x}{2} + \dfrac{4y}{2} = \dfrac{8}{2}$
Simplify:	$x + 2y = 4$

If this had been done before the equations were solved, then some of the working would have been slightly easier.

Notice from the working above that the effect of dividing both sides of the equation by 2 is that *each term* in the equation is divided by 2.

In fact, you can see that in general the following is true.

> Multiplying or dividing both sides of an equation by a number is the same as multiplying or dividing each term in the equation by the number.

This is known as *multiplying through* or *dividing through* by a number.

It's worth looking out for numbers that are common factors of the terms in an equation, and dividing through by them before you start working with the equation. Similarly, if an equation contains numerical fractions, then it's helpful to begin by multiplying through by a suitable integer to clear them. Doing these things can be particularly helpful when you use the elimination method for solving simultaneous equations, which you'll meet in the next subsection.

You should not divide an equation through by a *letter* that is a common factor of the terms, unless you know that the letter cannot be equal to zero.

Here's a summary of the method for solving simultaneous equations that you've seen in this subsection.

Strategy *To solve simultaneous equations: substitution method*

1. Rearrange one of the equations, if necessary, to obtain a formula for one unknown in terms of the other.
2. Use this formula to substitute for this unknown in the other equation.
3. You now have an equation in one unknown. Solve it to find the value of the unknown.
4. Substitute this value into an equation involving both unknowns to find the value of the other unknown.

(Check: Confirm that the two values satisfy the original equations.)

Try this strategy for yourself in the next activity.

Activity 27 *Solving more simultaneous equations by substitution*

Use the substitution method to solve the following pairs of simultaneous equations.

(a) $$\begin{aligned} A - 2B &= -3 \\ 2A + 3B &= 8 \end{aligned}$$

(b) $$\begin{aligned} 3S + T &= 3 \\ 7S + 2T &= 8 \end{aligned}$$

3.3 Elimination method for simultaneous equations

The method of Subsection 3.2 isn't always the simplest method. There's another method that you can use, and the idea behind it is as follows.

Suppose that you have a pair of simultaneous equations. Since the left-hand side of each equation is equal to its right-hand side, what you get by adding the two left-hand sides must be equal to what you get by adding the two right-hand sides.

For example, consider the following pair of simultaneous equations, which were solved earlier, in Example 15.

$$2x + 4y = 8,$$
$$-3x + 5y = -1.$$

If you add the left-hand sides and add the right-hand sides, then you obtain the equation

$$(2 - 3)x + (4 + 5)y = 8 - 1,$$

which simplifies to

$$-x + 9y = 7.$$

This equation must also hold for the two unknowns. We say that we have *added the two original equations.*

Adding these particular equations isn't very helpful! But adding other pairs of equations *can* be helpful, as you'll see in the next example.

Example 16 *Solving simultaneous equations by addition*

Tutorial clip

Solve the simultaneous equations

$$3A + 2B = 24, \quad (16)$$
$$3A - 2B = 36. \quad (17)$$

Solution

Notice that if you add the two equations, then the unknown B will be eliminated.

Adding equations (16) and (17) gives

$$(3 + 3)A + (2 - 2)B = 24 + 36$$
$$6A = 60$$
$$A = 10.$$

Substituting $A = 10$ into equation (16) gives

$$3 \times 10 + 2B = 24$$
$$30 + 2B = 24$$
$$2B = -6$$
$$B = -3.$$

So the solution is $A = 10$, $B = -3$.

(Check: Substituting $A = 10$, $B = -3$ into equation (16) gives

$$\text{LHS} = 3 \times 10 + 2 \times (-3) = 30 - 6 = 24 = \text{RHS}.$$

Substituting the same values into equation (17) gives

$$\text{LHS} = 3 \times 10 - 2 \times (-3) = 30 + 6 = 36 = \text{RHS}.)$$

You can also *subtract* two simultaneous equations. The reasoning behind this is the same: since the left-hand side of each equation is equal to its right-hand side, what you get by subtracting the two left-hand sides must be equal to what you get by subtracting the two right-hand sides.

In Example 16, the unknown B was eliminated by *adding* the two equations. This was possible because the coefficients of B in these equations have *the same value but with opposite signs*.

An alternative way to solve the simultaneous equations in Example 16 is to eliminate the other unknown, A, by *subtracting* the two equations. This is possible because the coefficients of A have *the same value with the same sign*. This way of solving the equations is shown in the next example.

Example 17 *Solving simultaneous equations by subtraction*

Solve the simultaneous equations

$$3A + 2B = 24, \quad (18)$$
$$3A - 2B = 36. \quad (19)$$

Solution

If you subtract one equation from the other, then the unknown A will be eliminated.

Subtracting equation (19) from (18) gives

$$(3-3)A + (2-(-2))B = 24 - 36$$
$$4B = -12$$
$$B = -3.$$

Substituting $B = -3$ into equation (18) gives

$$3A + 2 \times (-3) = 24$$
$$3A - 6 = 24$$
$$3A = 30$$
$$A = 10.$$

So the solution is $A = 10$, $B = -3$.

(This was checked in Example 16.)

The next activity gives you examples of each of the possibilities of adding and subtracting.

Activity 28 *Solving simultaneous equations by addition and by subtraction*

Use addition or subtraction to solve the following pairs of simultaneous equations.

(a) $4x - 5y = 7$
$-4x + 3y = -9$

(b) $5x + 4y = 23$
$5x + 6y = 27$

The pairs of simultaneous equations in Examples 16 and 17 and Activity 28 are unusual because a coefficient can be eliminated straightforwardly by addition or by subtraction. This is not possible for most pairs of simultaneous equations, but you can still use addition or subtraction to solve the equations – you just need another step first! This is illustrated in the next example.

Tutorial clip

Example 18 *Solving simultaneous equations by elimination*

Solve the simultaneous equations

$$3x + 2y = 8, \qquad (20)$$
$$-5x + 3y = -7. \qquad (21)$$

Solution

Here adding or subtracting the equations doesn't eliminate either x or y. However, you can make the coefficient of y the same in each equation by multiplying the first equation by 3 and the second equation by 2; then subtracting the new equations *will* eliminate y.

Multiplying equation (20) by 3 and equation (21) by 2 gives

$$9x + 6y = 24, \qquad (22)$$
$$-10x + 6y = -14. \qquad (23)$$

Multiplying both sides of a true equation by the same number results in another true equation; you saw this in Unit 5.

Now you can use subtraction.

Subtracting equation (23) from equation (22) gives

$$(9 - (-10))x + (6 - 6)y = 24 - (-14)$$
$$19x = 38$$
$$x = 2.$$

Substituting $x = 2$ into equation (20) gives

$$3 \times 2 + 2y = 8$$
$$6 + 2y = 8$$
$$2y = 2$$
$$y = 1.$$

Hence the solution is $x = 2$, $y = 1$.

(Check: Substituting $x = 2$, $y = 1$ into equation (20) gives

$$\text{LHS} = 3 \times 2 + 2 \times 1 = 6 + 2 = 8 = \text{RHS}.$$

Substituting the same values into equation (21) gives

$$\text{LHS} = -5 \times 2 + 3 \times 1 = -10 + 3 = -7 = \text{RHS}.)$$

Here's a summary of the method for solving simultaneous equations that you've seen in this section.

Strategy *To solve simultaneous equations: elimination method*

1. Multiply one or both of the equations by suitable numbers, if necessary, to obtain two equations that can be added or subtracted to eliminate one of the unknowns.
2. Add or subtract the equations to eliminate the unknown.
3. You now have an equation in one unknown. Solve it to find the value of the unknown.
4. Substitute this value into an equation involving both unknowns to find the value of the other unknown.

(Check: Confirm that the two values satisfy the original equations.)

Try this strategy for yourself in the next activity.

Activity 29 *Solving simultaneous equations by elimination*

Solve the simultaneous equations

$$2x + 3y = 9,$$
$$3x - 4y = 5.$$

You've now seen two strategies for solving simultaneous linear equations. Next there is an activity that gives you several pairs of simultaneous equations, and you can choose whichever method you prefer to solve each pair. Substitution is probably easier when one of the coefficients is 1 or -1, since then you can express one of the unknowns in terms of the other without having to introduce fractions. Elimination is probably easier otherwise. The solutions show how to use each method for each pair of equations, so if you would like some extra practice, then you can try both methods on each pair.

Remember that the solutions to simultaneous equations aren't necessarily integers! It's perfectly possible for them to be fractions, and this happens in parts (b) and (d) of the activity.

Activity 30 *Putting it all together*

Solve the following pairs of simultaneous equations.

(a) $2X - Y = -1$
$3X - Y = 1$

(b) $4X - 5Y = 1$
$5X + 2Y = 4$

(c) $-2A + 5B = -2$
$-2A + 3B = 2$

(d) $3P - 6Q = 3$
$7P - 4Q = 1$

Finally in this subsection, you may wonder if there's a straightforward way to tell whether a pair of equations is of the 'awkward' type that has either no solution or infinitely many solutions.

One way to tell is to rearrange both equations so that each is a formula for the same unknown in terms of the other unknown, and then use the facts in the box on page 148. You need to find out whether the equations represent parallel lines.

For example, consider the simultaneous equations

$$2X - Y = -1, \tag{24}$$
$$-4X + 2Y = 1. \tag{25}$$

Equation (24) can be rearranged to express Y in terms of X as follows:

$$\begin{aligned} 2X - Y &= -1 \\ 2X &= -1 + Y \\ 2X + 1 &= Y \\ Y &= 2X + 1. \end{aligned}$$

Similarly, equation (25) can be rearranged to express Y in terms of X as follows:

$$\begin{aligned} -4X + 2Y &= 1 \\ 2Y &= 4X + 1 \\ Y &= 2X + \tfrac{1}{2}. \end{aligned}$$

So the pair of equations is equivalent to the following pair:

$$\begin{aligned} Y &= 2X + 1, \\ Y &= 2X + \tfrac{1}{2}. \end{aligned}$$

You can now see that the two equations represent lines with the same gradient, 2, but not the same line. So the equations don't have a solution.

Actually, if you use the substitution or elimination method to try to solve a pair of equations that doesn't have a solution, then the problem shows up very quickly by giving a nonsense result. For example, multiplying equation (24) by 2 and leaving equation (25) unchanged gives

$$\begin{aligned} 4X - 2Y &= -2, \\ -4X + 2Y &= 1 \end{aligned}$$

and when you add these two equations, the result is

$$0 = -1.$$

This immediately tells you that (provided that *you* haven't made an arithmetical slip!) this pair of equations has no solution.

So far in this section, you have seen how to solve two simultaneous linear equations by either

- drawing the lines that represent the two equations on a graph and reading off the coordinates of the intersection point, or
- using algebra – the substitution method or the elimination method.

The advantage of using algebra is that if a solution exists, then you can calculate it exactly. It doesn't matter whether you use the substitution or elimination method – you can choose the method that you prefer or find easier. Remember, however, that not all simultaneous equations have exactly one solution: if the equations represent parallel lines, but not the same line, then they don't have a solution, and if they represent the same line, then they have infinitely many solutions.

3.4 Using simultaneous equations in real-life problems

In this subsection you'll see two scenarios where you can use simultaneous equations to solve real-life problems. Here's the first.

Choosing a venue

Suppose that a company wishes to hold an away weekend for staff development, and you've been asked to advise how to choose between two possible venues, the Sandmartin Hotel and the Swift Hotel.

The facilities offered at the two hotels are similar, so the most important factor to consider is the cost. Each hotel makes a charge for its conference rooms and a charge per person for accommodation, as shown in Table 1.

Table 1 Hotel charges

	Hire of conference rooms (£)	Accommodation per person (£)
Sandmartin Hotel	350	55
Swift Hotel	500	50

The Sandmartin charges less than the Swift for its conference rooms, but charges slightly more per person. So if there are only a few participants, then the Sandmartin will be less costly, whereas if many people attend, then it will be cheaper to book the Swift.

The boss of the company doesn't yet know how many participants there will be, but she would like some advice so that she can make a quick decision once she does know.

The first step in tackling this problem is to try to identify exactly what you need to do. As the number of participants increases, the cost of booking the Sandmartin will catch up with, and then overtake, the cost of booking the Swift. You need to find the number of participants where the overtaking happens, which is the number of participants for which the cost of the two hotels is the same. This number could turn out to be not an integer, but that doesn't matter: whatever it is, if the actual number of participants turns out to be less than that number, then the company boss should book the Sandmartin, whereas if it turns out to be more, then she should book the Swift.

The next step is to express what you know in terms of mathematics. You can use the information in Table 1 to find formulas for the cost of booking each hotel in terms of the number of participants. You need to choose letters to represent these two quantities, so let's use C to represent the cost in £ of the booking, and N to represent the number of participants.

Activity 31 *Finding the formulas*

Write down two formulas for C in terms of N, one for the Sandmartin Hotel and one for the Swift Hotel.

The formulas in the solution to Activity 31 are the equations of straight lines. To find the value of N for which each formula gives the same value of C, you need to find the point where the lines cross. You have seen two ways of doing this: a graphical method and an algebraic method. You are asked to use both methods in the next activity.

Graphplotter

Activity 32 *Solving the problem*

(a) Use the 'Two graphs' tab of Graphplotter to plot the graphs of the formulas for the costs of booking the two hotels, on the same axes. Graphplotter uses the letters x and y for the variables, so first you should rewrite the formulas that you found in Activity 31 with N replaced by x, and C replaced by y. Make sure that the axis scales extend far enough to show the crossing point – appropriate ranges are 0 to 60 for the x-axis and 0 to 3500 for the y-axis.

(b) Read off the coordinates of the crossing point of the two lines. What do these coordinates represent?

On Graphplotter you can zoom in to find coordinates more precisely, though this isn't necessary in this activity.

(c) Use algebra to solve the two simultaneous equations given by the formulas, and check that your answer agrees with your answer to part (b).

(d) How might you present the results found in this activity to the company boss?

In Activity 32 you used simultaneous equations to help with a choice between two options, each of which involves a fixed cost and a cost that depends on a variable quantity (the number of participants). This sort of choice crops up in many situations. For example, when you buy a mobile phone there's often an initial cost for the phone and a monthly cost for calls, texts and downloads. So the total cost of using a phone will be the initial cost plus the monthly cost times the number of months for which you use the phone, and the costs will be different for different phones and providers. Here the variable quantity is the number of months for which you use the phone. Similarly, when you are choosing which television to buy, you might consider not only the initial costs of the different models, but also the ongoing costs of the power that they consume.

The second scenario in this subsection involves an economic model.

Supply and demand

Most economic models assume that as the price of an item on the market goes up, the demand for the item decreases, because fewer people are willing to buy it. For example, you would expect fewer apples to be sold if the price is £3 per kg than if the price is £2 per kg. The simpler economic models assume that the relationship between price and demand is linear – that is, the graph of demand against price is a straight line.

It has been postulated that the demand for some goods actually goes up as a direct result of price rises. These goods are known as *Veblen goods*, after the economist and sociologist Thorstein Veblen (1857–1929). Possible Veblen goods might be expensive wines and perfumes, which could be bought at least in part to demonstrate how wealthy the buyer is.

It is also usually assumed that as the price of an item goes up, the supply increases, because there is a greater incentive for producers to make and sell the item. Again, the simple modelling assumption often made is that this relationship is linear.

Naturally, there are many other factors involved in the relationships between the price, demand and supply of a market item, such as seasonal factors in the case of foods, and political and ethical checks and balances to try to ensure general fairness. However, the simple linear models work reasonably well for items that are not particularly expensive and not central to people's lives.

The price at which the supply of an item is equal to its demand – that is, the price at which the suppliers are able to sell all that they have produced – is called the equilibrium price for the item. A person selling at below this price would probably be able to charge more and still sell all he could supply, so he would tend to increase his price; whereas a person selling at above this price would probably not be able to sell all that he could supply, so he would tend to decrease his price. The price would tend to move towards the equilibrium price, at which demand equals supply.

Figure 7 Braeburn apples

Suppose that data have been gathered on the supply and demand of Braeburn apples at various prices. Figure 8 shows the graph that results when the data points for supply against price, and for demand against price, are plotted on the same axes. Here P represents the price of the apples in pence per kg, and Q represents the quantity of apples supplied or demanded, in millions of kg. The regression lines for each of the two relationships have been drawn on the graph.

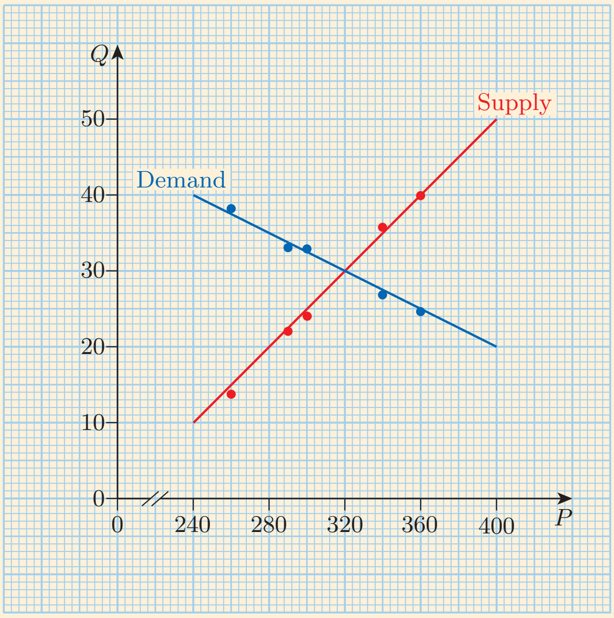


Figure 8 The supply and demand of Braeburn apples plotted against price

The equation of the regression line for the relationship between supply and price turns out to be

$$Q = \tfrac{1}{4}P - 50,$$

and the equation of the regression line for the relationship between demand and price turns out to be

$$Q = -\tfrac{1}{8}P + 70.$$

Activity 33 *Finding an equilibrium price*

In this activity you are asked to use the equations above to find the equilibrium price of Braeburn apples as follows.

(a) Multiply each of the equations

$$Q = \tfrac{1}{4}P - 50,$$
$$Q = -\tfrac{1}{8}P + 70$$

by a suitable integer to obtain two simultaneous equations that do not contain fractions.

(b) Solve the simultaneous equations that you obtained in part (a).

(c) State the equilibrium price of the apples. What is represented by the other value that you obtained as part of the solution?

In practice, an economic model like the one in Activity 33 would need to be adjusted for the effects of inflation, and other factors might need to be taken into account, as discussed earlier. The models are usually implemented using computer software, so that the calculations can be carried out automatically.

In Activity 33 you were asked to begin by multiplying the simultaneous equations by integers, to clear the fractions. As mentioned earlier, this is a useful technique to remember when you're dealing with simultaneous equations, or any other type of equations, that involve fractions.

Sometimes you can solve a practical problem by reading off an intersection point on a graph, even if you can't describe the graph using linear equations. You met an example of this in the video for Unit 2, which features a graph similar to the one shown in Figure 9. This graph shows the time that it takes to travel from Milton Keynes to Hemel Hempstead at different times of the day. One of the two curves shows the journey times for a route that uses a motorway, and the other curve shows the times for a route along main roads. By reading off the two intersection points, you can see that it is quicker to use the motorway between about 8 am and 7 pm, and quicker to use the main roads at other times.

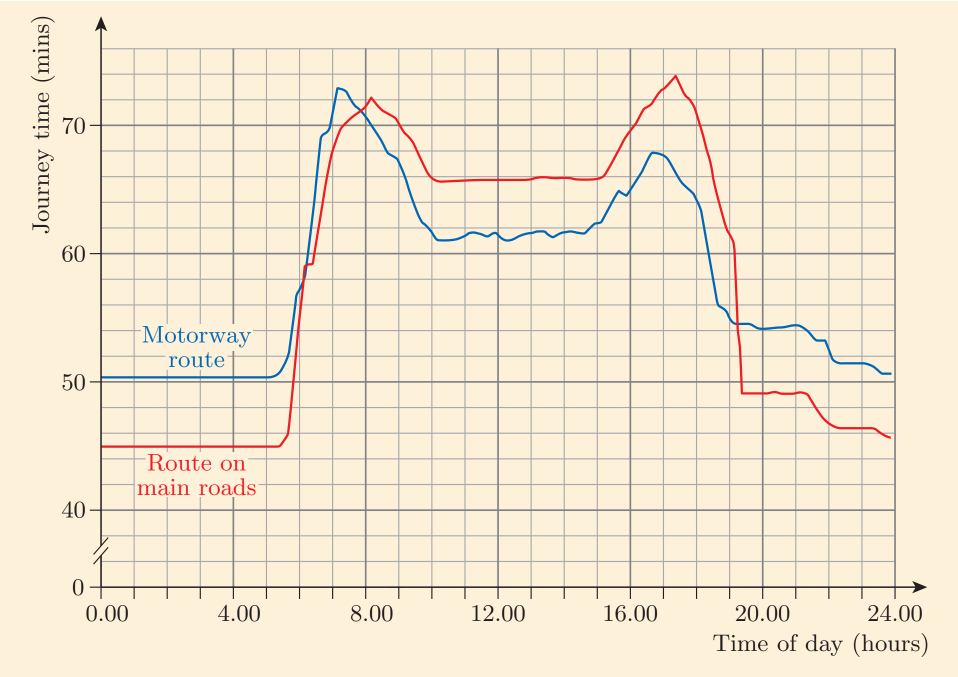


Figure 9 Journey times from Milton Keynes to Hemel Hempstead

3.5 More than two simultaneous equations

Both of the methods that you have seen for solving pairs of simultaneous equations in two unknowns involve an intermediate step of obtaining *one* equation in *one* unknown and solving that. This illustrates a very general principle, not only in mathematics, but in other branches of life.

When faced with an unfamiliar problem, see if you can turn it into a more familiar problem (so long as you know from experience that you can deal with the latter!). It's even worth turning the problem into *several* familiar problems, if that seems to be the only way forward.

So if you were faced with three equations in three unknowns, then you might expect that you could use one of the equations to reduce the other two to a pair of equations in *two* unknowns. This is indeed the case; consider, for example, the number puzzle in Figure 10 overleaf.

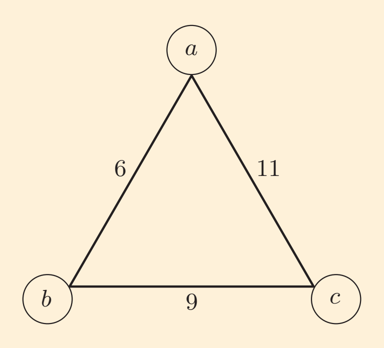


Figure 10 A number puzzle

The challenge is to replace a, b and c by three numbers in such a way that the number on each side of the triangle is the sum of the two numbers at the ends of that side. You might be able to work out the answer in your head, but using algebra definitely helps here! From the triangle, you can write down three equations in three unknowns, a, b and c:

$$a + c = 11, \tag{26}$$

$$a + b = 6, \tag{27}$$

$$b + c = 9. \tag{28}$$

If you subtract equation (27) from equation (26), you obtain

$$c - b = 5,$$

that is,

$$-b + c = 5, \tag{29}$$

You now have two equations, (28) and (29), in two unknowns, which you already know how to solve. Adding them together gives $2c = 14$, so $c = 7$. Substituting this value into equation (28) gives $b + 7 = 9$, so $b = 2$, and substituting it into equation (26) gives $a + 7 = 11$, so $a = 4$. So the puzzle is solved! This was done by reducing three equations in three unknowns to two equations in two unknowns.

It's worth knowing that being able to solve two equations in two unknowns puts you on the second step of an escalator that *in principle* allows you to deal with any number of the things!

Figure 11 Johann Carl Friedrich Gauss (1777–1855)

If you continue to study mathematics, then you will probably see an efficient formalisation of this process of solving simultaneous equations in many unknowns by reducing them to equations in progressively fewer unknowns. It is known as *Gaussian elimination* after the great eighteenth-century mathematician and physicist Johann Carl Friedrich Gauss.

In practice, these days most systems of several equations in several unknowns are solved by computer. The analysis of stress patterns in bridges and buildings, for instance, often requires systems of several thousand simultaneous equations.

4 Working with inequalities

All the examples that you have seen so far in this unit have concerned linear *equations*. Equality may not always be appropriate, though. If you need to arrive at an important interview on time, for example, then you may decide that you need to leave your house *at least* so many hours before the interview time, since you can drive on the motorway at a speed of *at most* 70 mph (if you are law-abiding). As you saw in Unit 2, facts like these can be expressed using *inequalities*.

In this section you'll see some more uses of inequalities, and how they can be manipulated in a similar way to equations.

4.1 Inequalities in one variable

In Unit 2 you saw how to express inequalities using the symbols below.

Inequality signs

$<$	is less than	$>$	is greater than
$\leq$	is less than or equal to	$\geq$	is greater than or equal to

In each of the next two activities you are asked to find an equality involving a variable, and write it down using one of the inequality signs above.

Activity 34 *Driving to an interview*

Suppose that you have to drive to an interview. It takes half an hour for you to reach the nearest motorway junction by car, and you must then drive 105 miles up the motorway. It's another quarter hour's drive from the motorway exit to where the interview takes place. Find an inequality for the length of time, t (in hours), that you must leave before the interview, assuming that you will obey the 70 mph motorway speed limit.

Since speed limits in the UK are given in miles per hour, and UK road maps normally give distances in miles, this activity uses miles rather than kilometres.

Activity 35 *Ordering CDs*

Suppose that you wish to order some CDs on the internet. You have up to £75 to spend, and the CDs are advertised at £6.50 each, with postage of £4 on an order of 15 or fewer CDs. Find an inequality concerning the number, n, of CDs that you may order.

Later in this section you'll see how to rearrange the inequality found in Activity 35 to help you work out the number of CDs that you can order.

Some of the examples that you saw in Section 2 have inequalities lurking under the surface. For instance, consider again the case of the group of friends travelling from one town to another, with Fred on his bicycle and the others in a car, which was discussed on page 143. Fred's equation, $d = 30t$, is valid only after the time $t = 0$ when he sets out, while the car's equation, $d = 90t - 135$, is valid only after the time $t = 1.5$ when *it* sets out. That is, Fred's equation is valid only when t is at least 0, and the car's equation is valid only when t is at least 1.5. These facts can be expressed as follows:

$$t \geq 0 \quad \text{for Fred's equation,}$$
$$t \geq 1.5 \quad \text{for the car's equation.}$$

Also, assuming that both the car and Fred stop when the car catches up, neither equation is valid after the time $t = 2.25$ when that happens. So t is at most 2.25 for each equation, and this fact can be expressed as

$$t \leq 2.25 \quad \text{for each equation.}$$

To sum up, for Fred's equation the variable t can take values only between 0 and 2.25, and for the car's equation the variable t can take values only between 1.5 and 2.25. As you saw in Unit 2, these facts can be expressed as *double inequalities*:

$$0 \leq t \leq 2.25 \quad \text{for Fred's equation,}$$
$$1.5 \leq t \leq 2.25 \quad \text{for the car's equation.}$$

4.2 Rearranging inequalities

In Activity 35, you should have found that the number n of CDs that you could buy must satisfy the inequality $6.5n + 4 \leq 75$. If you want to know how many CDs you can buy, then you need to rearrange this inequality into the form

$$n \leq \text{a number}.$$

This form tells you all the numbers that satisfy the inequality. Finding all the numbers that satisfy an inequality is known as **solving** the inequality.

Let's consider how you could rearrange inequalities. Once you have learned how to do this, you'll be asked to rearrange the CD inequality into the form above.

You've already learned how to rearrange *equations*: as you know, you can do any of the following things to a correct equation to obtain another correct equation:

- do the same thing to both sides
- simplify one or both sides
- swap the sides.

The things that you can do to both sides are: adding a number, subtracting a number, multiplying by a number and dividing by a non-zero number.

An inequality can be rearranged in much the same way; but there are differences.

In the first place, if you swap the sides of an inequality, then the sense of the inequality sign reverses. That is, $<$ becomes $>$, and $>$ becomes $<$. Similarly, $\leq$ becomes $\geq$, and $\geq$ becomes $\leq$. Of course, this is to be expected: since $1 < 2$, it follows automatically that $2 > 1$.

Simplifying one or both sides of an inequality is just as valid as simplifying one or both sides of an equation; for example, $4 > 1 + 2$ is correct, and its simplified form $4 > 3$ is also correct.

What about doing the same thing to both sides of an inequality, such as adding the same number to both sides, or multiplying both sides by the same number? The next activity lets you explore these possibilities.

Activity 36 *Doing the same thing to both sides of an inequality*

(a) Consider the correct inequality $-2 < 1$.

(i) Does it remain correct if you add 3 to both sides?

(ii) Does it remain correct if you subtract 4 from both sides?

(b) Consider the correct inequality $-3 \geq -5$.

(i) Does it remain correct if you multiply both sides by 3?

(ii) Does it remain correct if you multiply both sides by -2?

(c) Does each of the following correct inequalities remain correct if you multiply both sides by 0?

(i) $2 \leq 3$ (ii) $2 < 3$

The rearranged inequalities in Activity 36 illustrate the following facts.

Rearranging inequalities

You can do any of the following things to a correct inequality to obtain another correct inequality.

- Do any of the following to *both sides*.
  - Add or subtract a number.
  - Multiply or divide by a *positive* number.
  - Multiply or divide by a *negative* number, *if you reverse the inequality sign*.
- Simplify one side or both sides.
- Swap the sides, *if you reverse the inequality sign*.

You also saw in Activity 36 that if you multiply both sides of an inequality by zero, then both sides become 0, so if it was a '$\leq$' or a '$\geq$' inequality, then it remains correct, but if it was a '$<$' or a '$>$' inequality, it becomes wrong.

The next example illustrates how to rearrange inequalities. It's just like rearranging equations, except that sometimes you need to reverse the inequality sign, according to the rules in the box above.

Example 19 *Rearranging inequalities*

For each of the following inequalities, rearrange it to obtain an inequality with the variable by itself on the left-hand side and a number on the right-hand side, and illustrate the numbers that satisfy the original inequality on a number line.

(a) $3x + 2 < 1$ (b) $-3 - 2x \leq 1$

Solution

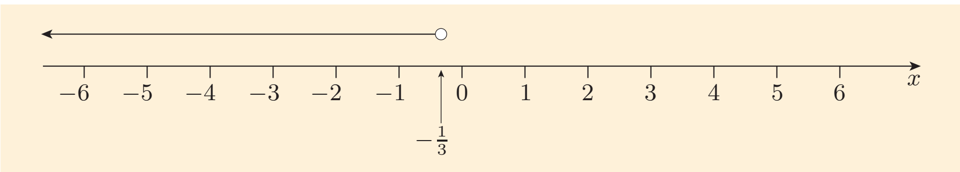
Use the usual method for solving a linear equation, remembering that sometimes you may need to reverse the inequality sign.

(a) The inequality is: $3x + 2 < 1$

Subtract 2: $3x < -1$

Divide by 3: $x < -\frac{1}{3}$

So the numbers that satisfy the original inequality can be illustrated on a number line as follows.



(b) The inequality is: $-3 - 2x \leq 1$

Add 3: $-2x \leq 4$

Divide by -2: $x \geq -2$

The inequality sign was reversed because -2 is negative.

Alternatively, you could proceed as follows.

The inequality is: $-3 - 2x \leq 1$

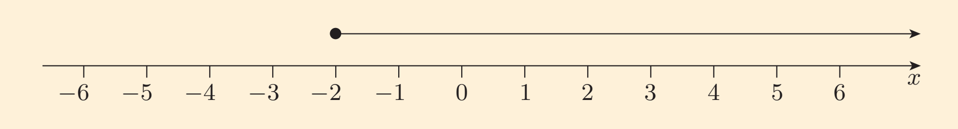
Add $2x$: $-3 \leq 1 + 2x$

Subtract 1: $-4 \leq 2x$

Divide by 2: $-2 \leq x$

Swap the sides: $x \geq -2$

So the numbers that satisfy the original inequality can be illustrated on a number line as follows.



When you rearrange an inequality, as when you rearrange an equation, there are often different but equally valid approaches that you can take. The approach that you choose will be valid as long as in each step you do the same thing to both sides, simplify one or both sides, or swap the sides, and you reverse the inequality sign when appropriate.

Activity 37 *Rearranging inequalities*

Rearrange each of the following inequalities to obtain an inequality with the variable by itself on the left-hand side and a number on the right-hand side, and illustrate the numbers that satisfy the inequality on a number line.

(a) $2z - 1 \geq 5$ (b) $7 - 3a < 1$ (c) $3 < -2p$

(d) $2 - 3t > t + 1$ (e) $\frac{1}{2}c - 1 \geq c$ (f) $m > 2(1 - m)$

Activity 38 *Rearranging another inequality*

In the solution to Activity 35, the inequality

$$6.5n + 4 \leq 75$$

was found for the number n of CDs that can be bought. Rearrange this inequality to obtain n by itself on the left-hand side and a number on the right-hand side. Hence state the maximum number of CDs that can be bought.

When you multiply or divide both sides of an inequality by a *variable*, or by an expression containing a variable, you have to be careful about the sense of the inequality sign. For example, consider the inequality

$$a < \frac{b}{c}.$$

If you know that c is positive, then you can clear the fraction by multiplying both sides by c and leaving the inequality sign unchanged. Similarly, if you know that c is negative, then you can multiply both sides by c and reverse the inequality sign. If you don't know whether c is positive or negative, then all you can do is to split into cases, like this:

If c is positive then $ac < b$, whereas if c is negative then $ac > b$.

However, inequalities like this don't arise often in practice, and you won't meet any more in this module.

4.3 Inequalities in two variables

In Unit 6 you saw that some equations in two variables can be represented by straight lines on graphs. In this section you'll see that some *inequalities*

in two variables can also be represented on graphs, and how it can be useful to do so.

Let's consider an inequality that arises from a practical situation. Suppose that you are organising a party, and you have £60 to spend on the soft drinks for it. You plan to buy some cartons of juice, which cost £1 each, and some bottles of lemonade, which cost £1.50 each. Let's use J and L to represent the numbers of cartons of juice and bottles of lemonade, respectively, that you buy. The total cost, in pounds, will be

$$J \times 1 + L \times 1.5 = J + 1.5L.$$

Since you have only £60 to spend,

$$J + 1.5L \leq 60.$$

So the numbers J and L of cartons of juice and bottles of lemonade that you buy must satisfy this inequality. It can be slightly easier to deal with whole numbers, so let's multiply both sides of this inequality by the positive number 2 to obtain the equivalent inequality

$$2J + 3L \leq 120.$$

Now let's look at how this inequality can be represented graphically.

First, consider the pairs of values of J and L such that $2J + 3L$ is not just *less than or equal* to 120, but actually *equal* to 120. That is, consider the pairs that satisfy the equation

$$2J + 3L = 120. \qquad (30)$$

These pairs correspond to spending *all* of the £60. You know how to draw the graph of equation (30) – it will be a straight line – so let's do that. We'll put J on the horizontal axis and L on the vertical axis. As you have seen, one way to draw a line is to find two points on it. Here, if $J = 0$, then

You saw on page 132 that any equation of the form $ax + by = d$, where x and y are variables and a, b and d are numbers, is the equation of a straight line.

$$3L = 120, \quad \text{so} \quad L = 40,$$

so one point on the line is $(0, 40)$. This corresponds to buying no cartons of juice and 40 bottles of lemonade. Similarly, if $L = 0$, then

$$2J = 120, \quad \text{so} \quad J = 60,$$

so another point on the line is $(60, 0)$. This corresponds to buying 60 cartons of juice and no bottles of lemonade. So the line is as shown in Figure 12.

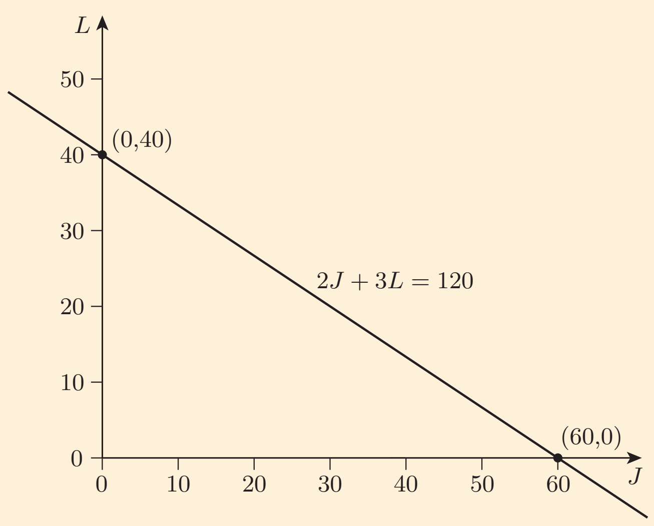


Figure 12 The straight line with equation $2J + 3L = 120$

Now consider a point somewhere on the line with equation $2J + 3L = 120$, say the point $(30, 20)$. The values $J = 30$, $L = 20$ also satisfy the inequality $2J + 3L \leq 120$, since 'less than or equal to' includes 'equal to'. Moreover, if either of the values $J = 30$ or $L = 20$ is *decreased* slightly, then the inequality will still hold. On the other hand, if either of the values is *increased*, to a value above 30 or 20, respectively, then $2J + 3L$ will exceed 120, and so the inequality will not be satisfied.

So the area below and to the left of the line $2J + 3L = 120$, *including* the line itself, represents the points that satisfy the inequality $2J + 3L \leq 120$, while the area above and to the right of the line, *but not including* the line, represents the points that do not satisfy the inequality. In Figure 13, the points that satisfy the inequality are those in the blue area or on the line, while the points that don't satisfy the inequality are those in the pink area.

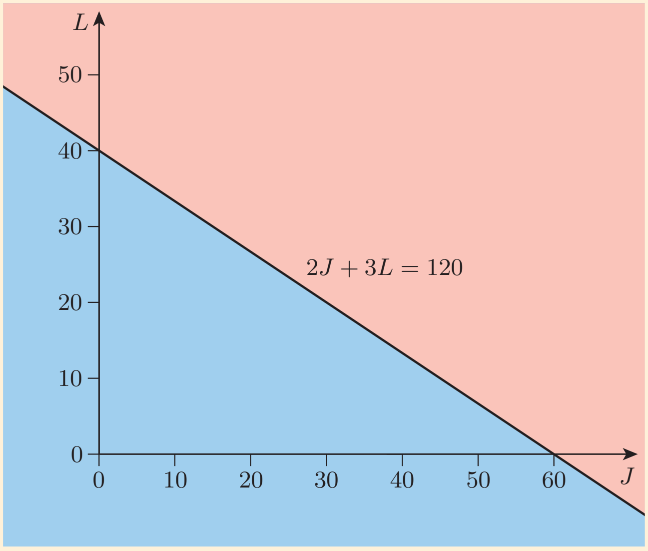


Figure 13 Points satisfying and not satisfying the inequality $2J + 3L \leq 120$

However, Figure 13 is not a particularly useful representation of the possibilities for buying drinks for the party, because it includes areas where one or both of J and L are *negative*. You can't buy negative quantities of cartons of juice or bottles of lemonade, so in order to model the choices for the party, it's necessary to consider only the area where J and L are positive. So the area of the graph that represents the practical choices for J and L is the blue triangle in Figure 14. More specifically, the practical choices are the points within this area that have *integer coordinates*.

For example, you could buy 10 cartons of juice and 30 bottles of lemonade; this comes well within the blue triangle in Figure 14. Or you could spend up to your £60 limit with 15 cartons of juice and 30 bottles of lemonade; the point on the graph that represents this expenditure is on the boundary of the blue triangle.

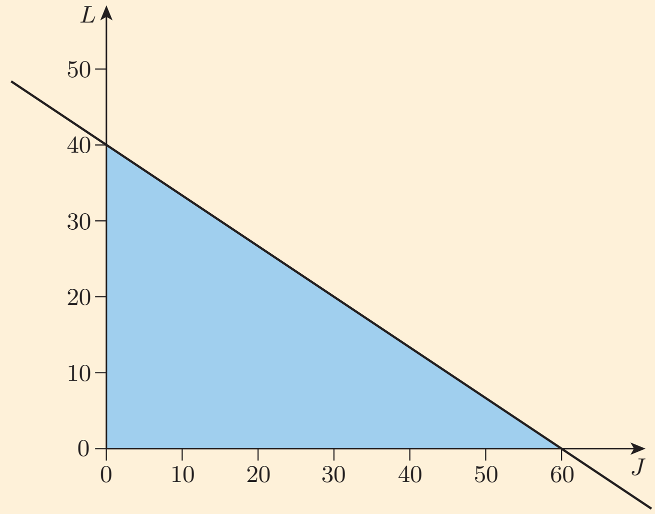


Figure 14 The area of practical choice

Activity 39 *Representing an inequality on a graph*

Now suppose that you wish to ensure that there is at least a litre of soft drink available to each party guest, of whom there will be 90. The juice cartons contain one litre each and the lemonade bottles contain three litres.

(a) Find an inequality in terms of J and L that expresses this wish.

(b) Draw a graph and shade in the area of practical choice for the inequality in part (a).

Finally, suppose that you want to choose values of J and L that satisfy *both* the original wish and the one in Activity 39. That is, you want to spend no more than £60 *and* you want to buy at least 90 litres of soft drink. Then the area of practical choice is the *overlap* of the two areas of practical choice drawn already, namely the area in Figure 14 and the area in the solution to Activity 39. This overlap is shown in Figure 15.

Graphs like the one in Figure 15 play a part in *operational research*. This is a branch of applied mathematics that is concerned with finding the best solutions to complicated practical situations.

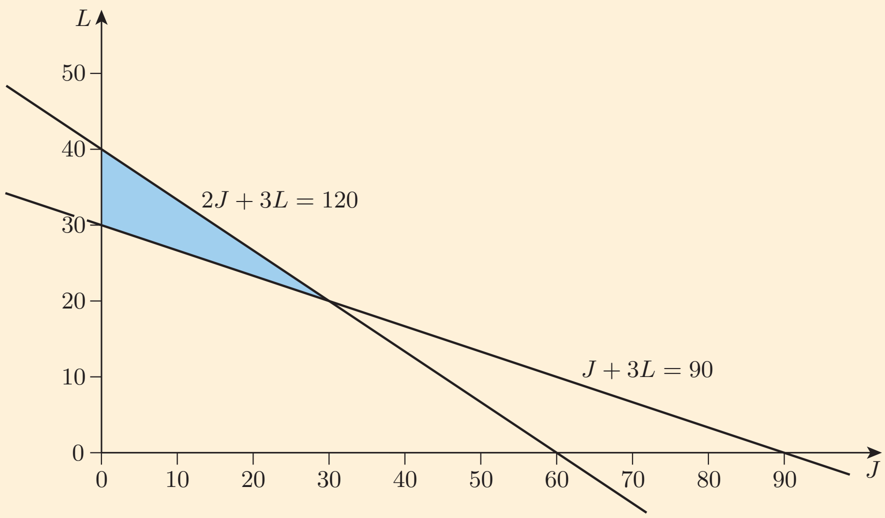


Figure 15 The area of practical choice for both inequalities together

Activity 40 *Choosing a suitable pair of values*

Use Figure 15 to find a suitable number of cartons of juice and a suitable number of bottles of lemonade that you could buy to ensure that you spend no more than £60 and buy at least 90 litres of soft drink.

In this unit you have seen various different situations in which algebra is helpful, and you have learned some new algebraic techniques that allow you to deal with situations of these types. In particular, you have learned to rearrange equations and inequalities, take out common factors and solve simultaneous linear equations.

Learning checklist

After studying this unit, you should be able to:

- rearrange a simple equation to make a chosen variable the subject
- recognise the equation of a straight line in different forms
- draw the graph of an equation of a straight line that is given in any form
- take common factors out of expressions, including highest common factors
- rearrange an equation to make a chosen variable the subject, where this involves taking out a common factor
- solve a pair of simultaneous linear equations in two unknowns
- understand how a pair of simultaneous linear equations in two unknowns is represented on a graph
- recognise when a pair of simultaneous equations in two unknowns has no solution or infinitely many solutions
- solve some practical problems involving simultaneous linear equations
- express information from a practical situation as an inequality
- rearrange an inequality
- interpret some inequalities in two unknowns as areas on graphs.

Solutions and comments on Activities

Activity 1

Substituting $f = 59$ into the formula

$$f = 1.8c + 32$$

gives

$$59 = 1.8c + 32.$$

Subtract 32: $27 = 1.8c$

Divide by 1.8: $15 = c$

So 59°F is the same as 15°C.

Activity 2

Substituting $f = 85.1$ into the formula

$$c = \frac{f - 32}{1.8}$$

gives

$$c = \frac{85.1 - 32}{1.8} = 29.5.$$

So 85.1°F is the same as 29.5°C.

Activity 3

The equation is: $f = 2s + 4$

Subtract 4: $f - 4 = 2s$

Divide by 2: $\dfrac{f-4}{2} = s$

Swap the sides: $s = \dfrac{f-4}{2}$

Activity 4

(a) The original formula is: $f = 1.8(k - 273) + 32$

Multiply out: $f = 1.8k - 491.4 + 32$

Simplify: $f = 1.8k - 459.4$

(This completes Step 1 of the strategy.)

Add 459.4: $f + 459.4 = 1.8k$

(This completes Step 2.)

Divide by 1.8: $\dfrac{f + 459.4}{1.8} = k$

(This completes Step 3.)

Hence a formula for k is

$$k = \frac{f + 459.4}{1.8}.$$

(You can obtain an alternative formula that does not contain the rather unmemorable number 459.4, by proceeding as follows.

The original formula is: $f = 1.8(k - 273) + 32$

Subtract 32: $f - 32 = 1.8(k - 273)$

Divide by 1.8: $\dfrac{f-32}{1.8} = k - 273$

Add 273: $\dfrac{f-32}{1.8} + 273 = k$

Hence a formula for k is

$$k = \frac{f-32}{1.8} + 273.$$

In this second manipulation, the strategy was not followed exactly, but at each step the same thing was done to each side, the sides were simplified or the sides were swapped, so the manipulation is valid. Later in the subsection there is a discussion about the fact that you do not need to follow the strategy exactly.)

(b) Substituting $f = 97.7$ into the formula found in part (a) gives

$$k = \frac{97.7 + 459.4}{1.8} = \frac{557.1}{1.8} = 310 \text{ (to 3 s.f.)}.$$

So 97.7°F is about the same as 310 K.

Activity 5

(a) The original formula is: $d = st$

Divide by s: $\dfrac{d}{s} = t$

Hence a formula for t is

$$t = \frac{d}{s}.$$

(b) Substituting $d = 96$ and $s = 80$ into the formula found in part (a) gives

$$t = \frac{96}{80} = 1.2.$$

Hence the time taken is 1.2 hours; that is, 1 hour and 12 minutes.

(0.2 of an hour is the same as 0.2×60 minutes, that is, 12 minutes.)

Activity 6

(a) The equation is: $y = \frac{x}{2} - 3$

Multiply by 2: $2y = 2\left(\frac{x}{2} - 3\right)$

Multiply out: $2y = x - 6$

Add 6: $2y + 6 = x$

Swap the sides: $x = 2y + 6$

(The sides were swapped because it is usual to write the subject on the left-hand side.)

(b) The equation is: $s = \frac{6}{r}$

Multiply by r: $rs = 6$

Divide by s (assuming that $s \neq 0$): $r = \frac{6}{s}$

(c) The equation is: $X = \frac{Z}{Y+1}$

Multiply by $Y + 1$: $X(Y + 1) = Z$

Swap the sides: $Z = X(Y + 1)$

(There's no need to multiply out the brackets, but it is okay to do so.)

(d) The equation is: $q = \frac{2}{3}(p + 2)$

Multiply by 3: $3q = 2(p + 2)$

Multiply out: $3q = 2p + 4$

Subtract 4: $3q - 4 = 2p$

Divide by 2: $\frac{3q - 4}{2} = p$

Swap the sides: $p = \frac{3q - 4}{2}$

(Instead of multiplying out the brackets above, you could have proceeded as follows.

$$3q = 2(p + 2)$$

Divide by 2: $\frac{3}{2}q = p + 2$

Subtract 2: $\frac{3}{2}q - 2 = p$

Swap the sides: $p = \frac{3}{2}q - 2$

As you can see, this is slightly more efficient, and it leads to a slightly different, but equivalent, formula.)

(e) The equation is: $c = \frac{b}{2d + 1}$

Multiply by $2d + 1$: $c(2d + 1) = b$

Multiply out: $2cd + c = b$

Subtract c: $2cd = b - c$

Divide by $2c$
(assuming that $c \neq 0$): $d = \frac{b - c}{2c}$

(Instead of multiplying out the brackets above, you could have proceeded as follows.

$$c(2d + 1) = b$$

Divide by c
(assuming that $c \neq 0$): $2d + 1 = \frac{b}{c}$

Subtract 1: $2d = \frac{b}{c} - 1$

Divide by 2: $d = \frac{\frac{b}{c} - 1}{2}$

Expand the fraction: $d = \frac{\left(\frac{b}{c}\right)}{2} - \frac{1}{2}$

Simplify: $d = \frac{b}{c} \times \frac{1}{2} - \frac{1}{2}$

Simplify more: $d = \frac{b}{2c} - \frac{1}{2}$

This leads to a slightly different, but equivalent, formula.)

Activity 7

(a) Substituting $x = 3$ into the equation $y = 3x - 2$ gives

$$y = 3 \times 3 - 2 = 9 - 2 = 7.$$

So the point $(3, 7)$ satisfies the equation.

(b) Substituting $x = 3$ and $y = 7$ into the equation $y + 2 = 3x$ gives

$$\text{LHS} = 7 + 2 = 9$$

and

$$\text{RHS} = 3 \times 3 = 9.$$

Since LHS = RHS, the point $(3, 7)$ satisfies the equation.

(When you check whether an equation is satisfied, it is usually best to substitute into each side separately, as in the solution to part (b). However, as you have seen, if one side of the equation is very simple, such as a number or letter by itself, then you can just substitute into the other side and confirm that you get the expected answer, as in part (a).)

Activity 8

The equation is: $3x + 2y = 6$

Subtract $3x$: $2y = -3x + 6$

Divide by 2: $y = \frac{-3x + 6}{2}$

Expand the fraction: $y = \frac{-3x}{2} + \frac{6}{2}$

Simplify: $y = -\frac{3}{2}x + 3$

The gradient is $-\frac{3}{2}$ and the y-intercept is 3.

One point on the line is $(0, 3)$ (the point corresponding to the y-intercept). Another point can be found by substituting $x = 1$, for example, into the equation

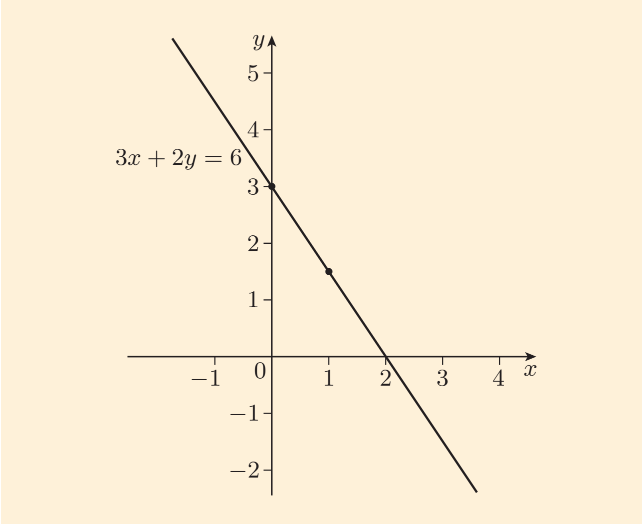
$$y = -\tfrac{3}{2}x + 3.$$

This gives

$$y = -\tfrac{3}{2} \times 1 + 3 = -\tfrac{3}{2} + 3 = \tfrac{3}{2}.$$

So another point is $(1, \frac{3}{2}) = (1, 1.5)$.

The line is shown on the next page.



Activity 9

Substituting $x = 0$ into the equation gives $-3y = -6$.

Divide by -3: $y = 2$

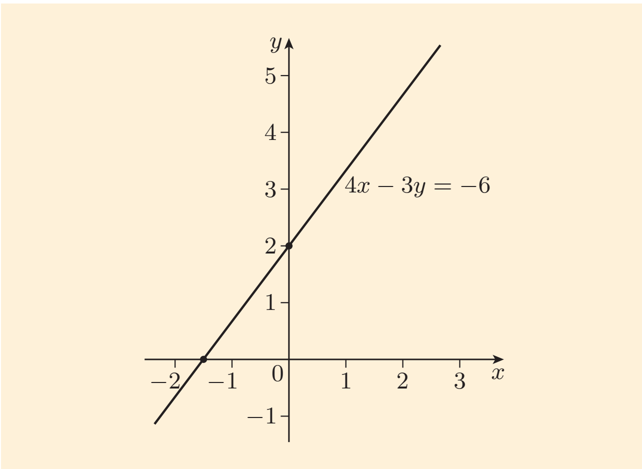
So a point on the line is $(0, 2)$.

Substituting $y = 0$ into the equation gives $4x = -6$.

Divide by 4: $x = -\frac{3}{2}$.

So another point on the line is $\left(-\frac{3}{2}, 0\right) = (-1.5, 0)$.

The line is shown below.



Activity 10

(a) $pqr = q \times pr$

(b) $A^7 = A^4 \times A^3$

(c) $4f^3 = 2f \times 2f^2$

(d) $p^3q^5 = p^2q^2 \times pq^3$

(e) $6x^5y^8 = 2x^2y \times 3x^3y^7$

Activity 11

(a) $2z = z \times 2$ and $z^2 = z \times z$.

So z is a common factor of the two terms.

(b) $p^2q^2 = p^2 \times q^2$ and $p^2 = p^2 \times 1$.

So p^2 is a common factor of the two terms.

(c) $2A^2B^2 = 2AB \times AB$, $4A^2B = 2AB \times 2A$ and $8AB = 2AB \times 4$.

So $2AB$ is a common factor of the three terms.

Activity 12

(a) The highest common factor of $2ab^2$ and $4ab$ is $2ab$.

$2ab^2 = 2ab \times b$ and $4ab = 2ab \times 2$.

(b) The highest common factor of $3xy$ and $6y$ is $3y$.

$3xy = 3y \times x$ and $6y = 3y \times 2$.

(c) The highest common factor of $4p^3$, $9p^2$ and $2p^5$ is p^2.

$4p^3 = p^2 \times 4p$, $9p^2 = p^2 \times 9$ and $2p^5 = p^2 \times 2p^3$.

(d) The highest common factor of $10r$ and $15s$ is 5.

$10r = 5 \times 2r$ and $15s = 5 \times 3s$.

Activity 13

(a) $ab + a^2 = a \times b + a \times a = a(b + a)$

(b) $x^3y + yz = y \times x^3 + y \times z = y(x^3 + z)$

(c) $2w^2 + w^3 = w^2 \times 2 + w^2 \times w = w^2(2 + w)$

(d) $2z + 6z^4 = 2z \times 1 + 2z \times 3z^3 = 2z(1 + 3z^3)$

Activity 14

(a) $$2ab + 2b - 6b^2 = 2b \times a + 2b \times 1 - 2b \times 3b$$
$$= 2b(a + 1 - 3b)$$

(b) $A^5 - A^4 = A^4 \times A - A^4 \times 1 = A^4(A - 1)$

Activity 15

(a) $ab - 9bc = b(a - 9c)$

(b) $x^2 - x^5 + 2x^3 = x^2(1 - x^3 + 2x)$

(c) $-2rs + 4r^2s^2 = 2rs(-1 + 2rs)$

(d) $x\sqrt{y} - \sqrt{y} = \sqrt{y}(x - 1)$

Activity 16

(a) $12u + 6u^3 - 9u^2 = 3u(4 + 2u^2 - 3u)$

(b) $5r^2 - 10 = 5(r^2 - 2)$

(c) The terms of the expression $3fg - 2gh + 6fh$ have no common factors.

(d) $-8ABC - 4AB^2 + 2AB$
$= 2AB(-4C - 2B + 1)$

Activity 17

(a) $0.3m^2 - 0.6m + 0.9 = 0.3(m^2 - 2m + 3)$

(b) $\frac{1}{2}x - \frac{1}{2}x^2 = \frac{1}{2}x(1-x)$

Activity 18

(a) $$-2u^2 - 2u^3 - 4u^4 = -(2u^2 + 2u^3 + 4u^4)$$
$$= -2u^2(1 + u + 2u^2)$$

(b) $-1 - a + a^2 = -(1 + a - a^2)$

(c) $$pq - p^2q - q^2p - p^2q^2$$
$$= -(-pq + p^2q + q^2p + p^2q^2)$$
$$= -pq(-1 + p + q + pq)$$
$$= -pq(p + q + pq - 1)$$

(In the final step of part (c), the order of the terms inside the brackets has been changed, so that the term with a minus sign is not first. This is not essential, but it makes the expression slightly shorter and tidier.)

Activity 19

(a) The equation is: $gh = g + h$

Subtract h: $gh - h = g$

Take out h as a common factor: $h(g-1) = g$

Divide by $g - 1$ (assuming that $g \neq 1$): $h = \dfrac{g}{g-1}$

(b) The equation is: $x = \dfrac{x-y}{y}$

Multiply by y: $xy = x - y$

Add y: $xy + y = x$

Take out y as a common factor: $y(x+1) = x$

Divide by $x + 1$ (assuming that $x \neq -1$): $y = \dfrac{x}{x+1}$

(c) The equation is: $r = \dfrac{s}{r} + 2s$

Multiply by r: $r^2 = r\left(\dfrac{s}{r} + 2s\right)$

Multiply out the brackets: $r^2 = s + 2rs$

Take out s as a common factor: $r^2 = s(1 + 2r)$

Divide by $1 + 2r$ (assuming that $r \neq -\frac{1}{2}$): $\dfrac{r^2}{1+2r} = s$

Swap the sides: $s = \dfrac{r^2}{1+2r}$

(d) The equation is: $2p + q = r(p+q)$

Multiply out the brackets: $2p + q = rp + rq$

Subtract rp: $2p + q - rp = rq$

Subtract q: $2p - rp = rq - q$

Take out p as a common factor: $p(2 - r) = rq - q$

Divide by $2 - r$ (assuming that $r \neq 2$): $p = \dfrac{rq - q}{2 - r}$

This equation can be expressed slightly more simply if you take out the common factor in the numerator:

$$p = \frac{q(r-1)}{2-r}.$$

Activity 20

(a) The cost of hiring the venue is £100, and the cost of refreshments is £$10n$, so the total cost of the party, in pounds, is

$$100 + 10n.$$

The number of paying partygoers is $n - 5$. Hence the amount £p paid by each paying partygoer is given by the formula

$$p = \frac{100 + 10n}{n - 5}.$$

(b) The equation is: $p = \dfrac{100 + 10n}{n-5}$

Multiply by $n - 5$: $p(n-5) = 100 + 10n$

(This removes the fraction.)

Multiply out the brackets: $np - 5p = 100 + 10n$

Subtract $10n$: $np - 5p - 10n = 100$

Add $5p$: $np - 10n = 100 + 5p$

Take out n as a common factor: $n(p - 10) = 100 + 5p$

Divide by $p - 10$ (assuming that $p \neq 10$): $n = \dfrac{100 + 5p}{p - 10}$

(c) We substitute $p = 12.5$ into the formula. This gives

$$n = \frac{100 + 5 \times 12.5}{12.5 - 10}$$
$$= \frac{100 + 62.5}{2.5}$$
$$= \frac{162.5}{2.5}$$
$$= 65.$$

The number of partygoers who must attend is 65.

Activity 21

(a) The total cost of manufacturing the n games, in pounds, is $f + nc$.

(b) The total selling price of the n games, in pounds, is ns.

(c) The profit is given by

$$p = ns - (f + nc),$$

that is,

$$p = ns - f - nc.$$

(d)

The equation is:	$p = ns - f - nc$
Add f:	$p + f = ns - nc$
Swap the sides:	$ns - nc = p + f$
Take out n as a common factor:	$n(s - c) = p + f$
Divide by $s - c$ (valid if $s \neq c$):	$n = \dfrac{p + f}{s - c}$

(e) Substituting $p = 20\,000$, $f = 10\,000$, $s = 40$ and $c = 15$ into the formula gives

$$n = \frac{20\,000 + 10\,000}{40 - 15} = \frac{30\,000}{25} = 1200.$$

So the number of games required is 1200.

Activity 22

(a) From the graph it looks as if the car will catch up with Fred about 2.25 hours after Fred sets off, which is 2 hours and 15 minutes after he sets off. Since Fred will set off at 10 am, this will be at about quarter past 12.

(b) From the graph it looks as if the car will catch up with the bicycle after about 68 km.

Activity 23

(a)

The equation is:	$30t = 90t - 135$
Subtract $30t$:	$0 = 60t - 135$
Add 135:	$135 = 60t$
Divide by 60:	$2.25 = t$

Thus the solution is $t = 2.25$.

(b) Equation (6) is

$$d = 30t.$$

Substituting $t = 2.25$ into this equation gives

$$d = 30 \times 2.25 = 67.5.$$

(c) The car will catch up with the bicycle after 2.25 hours, which is 2 hours and 15 minutes. This will be at 12.15 pm. At this time the car and the bicycle will be 67.5 km from the start point.

(Notice that these values are close to the approximate values obtained from the graph.)

(d) Substituting $t = 2.25$ and $d = 67.5$ into equation (6) gives

$$\text{LHS} = 67.5,$$
$$\text{RHS} = 30 \times 2.25 = 67.5.$$

Substituting the same values into equation (7) gives

$$\text{LHS} = 67.5,$$
$$\text{RHS} = 90 \times 2.25 - 135 = 202.5 - 135 = 67.5.$$

Thus both equations are satisfied.

Activity 24

The equations are

$$y = 2x - 3, \tag{31}$$
$$y = 5x + 9. \tag{32}$$

The right-hand sides must be equal to each other; that is,

$$2x - 3 = 5x + 9.$$

This equation can be solved as follows.

The equation is:	$2x - 3 = 5x + 9$
Subtract $2x$:	$-3 = 3x + 9$
Subtract 9:	$-12 = 3x$
Divide by 3:	$-4 = x$

So $x = -4$. Substituting this value of x into equation (31) gives

$$y = 2 \times (-4) - 3 = -8 - 3 = -11.$$

(Check: Substituting $x = -4$, $y = -11$ into equation (31) gives

$$\text{LHS} = -11,$$
$$\text{RHS} = 2 \times (-4) - 3 = -11.$$

Substituting the same values into equation (32) gives

$$\text{LHS} = -11,$$
$$\text{RHS} = 5 \times (-4) + 9 = -11.)$$

Activity 25

(a) The equations

$$y = -2x + 5,$$
$$y = -2x - 1$$

represent parallel lines, so they do not have a solution.

(b) The equations can be arranged to give

$$y = -2x + 5,$$
$$y = -2x + 5.$$

As they represent the same line, they have infinitely many solutions.

(c) The equations

$$y = -2x + 5,$$
$$y = 3x - 1$$

represent lines that are not parallel, so they have exactly one solution.

Activity 26

The equations are

$$x = 2y - 7, \qquad (33)$$

$$3x - y = -6. \qquad (34)$$

Using equation (33) to substitute for x in equation (34) gives

$$3(2y - 7) - y = -6.$$

We now solve this equation.

$$3(2y - 7) - y = -6$$
$$6y - 21 - y = -6$$
$$5y - 21 = -6$$
$$5y = 15$$
$$y = 3$$

Substituting $y = 3$ into equation (33) gives

$$x = 2 \times 3 - 7 = 6 - 7 = -1.$$

So the solution is

$$x = -1, \ y = 3.$$

(Check: Substituting $x = -1$, $y = 3$ into equation (34) gives

$$\text{LHS} = 3 \times (-1) - 3 = -3 - 3 = -6 = \text{RHS.})$$

Activity 27

(a) The equations are

$$A - 2B = -3, \qquad (35)$$

$$2A + 3B = 8. \qquad (36)$$

Making A the subject of equation (35) gives

$$A = 2B - 3. \qquad (37)$$

Substituting for A in equation (36) gives

$$2(2B - 3) + 3B = 8$$
$$4B - 6 + 3B = 8$$
$$7B - 6 = 8$$
$$7B = 14$$
$$B = 2.$$

Substituting for B in equation (37) gives

$$A = 2 \times 2 - 3 = 4 - 3 = 1.$$

Thus the solution is

$$A = 1, \ B = 2.$$

(Check: Substituting $A = 1$, $B = 2$ into equation (35) gives

$$\text{LHS} = 1 - 2 \times 2 = 1 - 4 = -3 = \text{RHS}.$$

Substituting the same values into equation (36) gives

$$\text{LHS} = 2 \times 1 + 3 \times 2 = 2 + 6 = 8 = \text{RHS.})$$

(b) The equations are

$$3S + T = 3, \qquad (38)$$

$$7S + 2T = 8. \qquad (39)$$

Making T the subject of equation (38) gives

$$T = 3 - 3S. \qquad (40)$$

Substituting for T in equation (39) gives

$$7S + 2(3 - 3S) = 8$$
$$7S + 6 - 6S = 8$$
$$S + 6 = 8$$
$$S = 2.$$

Substituting for S in equation (40) gives

$$T = 3 - 3 \times 2 = 3 - 6 = -3.$$

Thus the solution is

$$S = 2, \ T = -3.$$

(Check: Substituting $S = 2$, $T = -3$ into equation (38) gives

$$\text{LHS} = 3 \times 2 + (-3) = 6 - 3 = 3 = \text{RHS}.$$

Substituting the same values into equation (39) gives

$$\text{LHS} = 7 \times 2 + 2 \times (-3) = 14 - 6 = 8 = \text{RHS.})$$

Activity 28

(a) The equations are

$$4x - 5y = 7, \qquad (41)$$

$$-4x + 3y = -9. \qquad (42)$$

Adding equations (41) and (42) gives

$$(4 - 4)x + (-5 + 3)y = 7 - 9$$
$$-2y = -2$$
$$y = 1.$$

Substituting this value of y into equation (41) gives

$$4x - 5 \times 1 = 7$$
$$4x - 5 = 7$$
$$4x = 12$$
$$x = 3.$$

Thus the solution is $x = 3$, $y = 1$.

(Check: Substituting $x = 3$, $y = 1$ into equation (41) gives

$$\text{LHS} = 4 \times 3 - 5 \times 1 = 12 - 5 = 7 = \text{RHS}.$$

Substituting the same values into equation (42) gives

$$\text{LHS} = -4 \times 3 + 3 \times 1 = -12 + 3 = -9 = \text{RHS.})$$

(b) The equations are

$$5x + 4y = 23, \qquad (43)$$

$$5x + 6y = 27. \qquad (44)$$

Subtracting equation (44) from equation (43) gives

$$(5 - 5)x + (4 - 6)y = 23 - 27$$
$$-2y = -4$$
$$y = 2.$$

(Instead of subtracting equation (44) from equation (43) you could have subtracted

equation (43) from equation (44), which gives

$$2y = 4$$
$$y = 2.)$$

Substituting this value of y into equation (43) gives

$$5x + 4 \times 2 = 23$$
$$5x + 8 = 23$$
$$5x = 15$$
$$x = 3.$$

Thus the solution is $x = 3$, $y = 2$.

(Check: Substituting $x = 3$, $y = 2$ into equation (43) gives

$$\text{LHS} = 5 \times 3 + 4 \times 2 = 15 + 8 = 23 = \text{RHS}.$$

Substituting the same values into equation (44) gives

$$\text{LHS} = 5 \times 3 + 6 \times 2 = 15 + 12 = 27 = \text{RHS}.)$$

Activity 29

The equations are

$$2x + 3y = 9, \qquad (45)$$
$$3x - 4y = 5. \qquad (46)$$

Probably the simplest choice is to multiply equation (45) by 3 and equation (46) by 2 to obtain

$$6x + 9y = 27, \qquad (47)$$
$$6x - 8y = 10. \qquad (48)$$

Subtracting equation (48) from equation (47) gives

$$17y = 17$$
$$y = 1.$$

Substituting $y = 1$ into equation (45) gives

$$2x + 3 \times 1 = 9$$
$$2x + 3 = 9$$
$$2x = 6$$
$$x = 3.$$

Thus the solution to equations (45) and (46) is $x = 3$, $y = 1$.

(Check: Substituting $x = 3$, $y = 1$ into equation (45) gives

$$\text{LHS} = 2 \times 3 + 3 \times 1 = 6 + 3 = 9 = \text{RHS}.$$

Substituting the same values into equation (46) gives

$$\text{LHS} = 3 \times 3 - 4 \times 1 = 9 - 4 = 5 = \text{RHS}.)$$

Activity 30

(a) The equations are

$$2X - Y = -1, \qquad (49)$$
$$3X - Y = 1. \qquad (50)$$

Substitution method

Equation (49) can be rearranged as

$$Y = 2X + 1. \qquad (51)$$

Substituting into equation (50) gives

$$3X - (2X + 1) = 1$$
$$3X - 2X - 1 = 1$$
$$X - 1 = 1$$
$$X = 2.$$

Substituting into equation (51) gives

$$Y = 2 \times 2 + 1 = 5$$

So the solution is $X = 2$, $Y = 5$.

Elimination method

Subtracting equation (49) from equation (50) gives

$$X = 2.$$

Substituting into equation (49) gives

$$2 \times 2 - Y = -1$$
$$4 - Y = -1$$
$$-Y = -5$$
$$Y = 5.$$

So the solution is $X = 2$, $Y = 5$.

(Check: Substituting $X = 2$, $Y = 5$ into equation (49) gives

$$\text{LHS} = 2 \times 2 - 5 = -1 = \text{RHS}.$$

Substituting the same values into equation (50) gives

$$\text{LHS} = 3 \times 2 - 5 = 1 = \text{RHS}.)$$

(b) The equations are

$$4X - 5Y = 1, \qquad (52)$$
$$5X + 2Y = 4. \qquad (53)$$

Substitution method

Rearranging equation (53) gives

$$5X + 2Y = 4$$
$$2Y = 4 - 5X$$
$$Y = 2 - \tfrac{5}{2}X. \qquad (54)$$

Substituting into equation (52) gives

$$4X - 5\left(2 - \tfrac{5}{2}X\right) = 1$$
$$4X - 10 + \tfrac{25}{2}X = 1$$
$$\tfrac{33}{2}X = 11$$
$$X = \tfrac{2}{3}.$$

Substituting into equation (54) gives

$$Y = 2 - \tfrac{5}{2} \times \tfrac{2}{3} = \tfrac{12}{6} - \tfrac{10}{6} = \tfrac{2}{6} = \tfrac{1}{3}.$$

So the solution is $X = \frac{2}{3}$, $Y = \frac{1}{3}$.

Elimination method

Multiplying equation (52) by 2 and equation (53) by 5 gives

$$8X - 10Y = 2,$$
$$25X + 10Y = 20.$$

Adding the two equations gives

$$33X = 22$$
$$X = \tfrac{2}{3}.$$

Substituting into equation (52) gives

$$4 \times \tfrac{2}{3} - 5Y = 1$$
$$\tfrac{8}{3} - 5Y = 1$$
$$\tfrac{8}{3} = 1 + 5Y$$
$$\tfrac{5}{3} = 5Y$$
$$\tfrac{1}{3} = Y.$$

So the solution is $X = \frac{2}{3}$, $Y = \frac{1}{3}$.

(Check: Substituting $X = \frac{2}{3}, Y = \frac{1}{3}$ into equation (52) gives

$$\text{LHS} = 4 \times \tfrac{2}{3} - 5 \times \tfrac{1}{3} = \tfrac{8}{3} - \tfrac{5}{3} = \tfrac{3}{3} = 1 = \text{RHS}.$$

Substituting the same values into equation (53) gives

$$\text{LHS} = 5 \times \tfrac{2}{3} + 2 \times \tfrac{1}{3} = \tfrac{10}{3} + \tfrac{2}{3} = \tfrac{12}{3} = 4 = \text{RHS.})$$

(c) The equations are

$$-2A + 5B = -2, \qquad (55)$$
$$-2A + 3B = 2. \qquad (56)$$

Substitution method

Rearranging equation (56) gives

$$-2A + 3B = 2$$
$$-2A = -3B + 2$$
$$A = \tfrac{3}{2}B - 1. \qquad (57)$$

Substituting into equation (55) gives

$$-2\left(\tfrac{3}{2}B - 1\right) + 5B = -2$$
$$-3B + 2 + 5B = -2$$
$$2B + 2 = -2$$
$$2B = -4$$
$$B = -2.$$

Substituting into equation (57) gives

$$A = \tfrac{3}{2} \times (-2) - 1 = -3 - 1 = -4.$$

So the solution is $A = -4$, $B = -2$.

Elimination method

Subtracting equation (56) from equation (55) gives

$$2B = -4$$
$$B = -2.$$

Substituting into equation (55) gives

$$-2A + 5 \times (-2) = -2$$
$$-2A - 10 = -2$$
$$-2A = 8$$
$$A = -4.$$

So the solution is $A = -4$, $B = -2$.

(Check: Substituting $A = -4$, $B = -2$ into equation (55) gives

$$\text{LHS} = -2 \times (-4) + 5 \times (-2) = 8 - 10 = -2$$
$$= \text{RHS}.$$

Substituting the same values into equation (56) gives

$$\text{LHS} = -2 \times (-4) + 3 \times (-2) = 8 - 6 = 2$$
$$= \text{RHS.})$$

(d) The equations are

$$3P - 6Q = 3,$$
$$7P - 4Q = 1.$$

Dividing the first equation by the common factor 3 gives the equations

$$P - 2Q = 1, \qquad (58)$$
$$7P - 4Q = 1. \qquad (59)$$

Substitution method

Equation (58) can be rearranged as

$$P = 2Q + 1. \qquad (60)$$

Substituting into equation (59) gives

$$7(2Q + 1) - 4Q = 1$$
$$14Q + 7 - 4Q = 1$$
$$10Q + 7 = 1$$
$$10Q = -6$$
$$Q = -\tfrac{3}{5}.$$

Substituting into equation (60) gives

$$P = 2 \times \left(-\tfrac{3}{5}\right) + 1 = -\tfrac{6}{5} + \tfrac{5}{5} = -\tfrac{1}{5}.$$

So the solution is $P = -\frac{1}{5}$, $Q = -\frac{3}{5}$.

Elimination method

Multiplying equation (58) by 2 and leaving equation (59) unchanged gives

$$2P - 4Q = 2,$$
$$7P - 4Q = 1.$$

Subtracting the first of these equations from the second gives

$$5P = -1$$
$$P = -\tfrac{1}{5}.$$

Substituting into equation (58) gives

$$-\tfrac{1}{5} - 2Q = 1$$
$$-2Q = \tfrac{1}{5} + 1$$
$$-2Q = \tfrac{6}{5}$$
$$Q = -\tfrac{3}{5}.$$

So the solution is $P = -\frac{1}{5}$, $Q = -\frac{3}{5}$.

(Check: Substituting $P = -\frac{1}{5}$, $Q = -\frac{3}{5}$ into equation (58) gives

$$\text{LHS} = -\tfrac{1}{5} - 2 \times (-\tfrac{3}{5}) = -\tfrac{1}{5} + \tfrac{6}{5} = 1 = \text{RHS}.$$

Substituting the same values into equation (59) gives

$$\text{LHS} = 7 \times (-\tfrac{1}{5}) - 4 \times (-\tfrac{3}{5}) = -\tfrac{7}{5} + \tfrac{12}{5} = 1$$
$$= \text{RHS}.)$$

Activity 31

The Sandmartin Hotel charges £350 plus £55 for each person, so its formula is

$$C = 350 + 55N.$$

The Swift Hotel charges £500 plus £50 for each person, so its formula is

$$C = 500 + 50N.$$

Activity 32

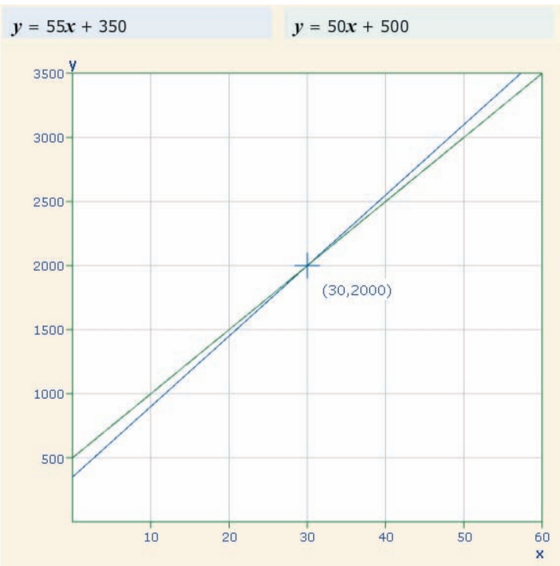
(a) Replacing N by x and C by y in the formula found in Activity 31 gives

$$y = 350 + 55x, \quad \text{that is,} \quad y = 55x + 350,$$

for the Sandmartin Hotel and

$$y = 500 + 50x, \quad \text{that is,} \quad y = 50x + 500,$$

for the Swift Hotel.



(b) The coordinates of the crossing point seem to be about $(30, 2000)$. The x-coordinate of this point represents the number of participants for which the costs of the two hotels are the same. The y-coordinate represents the cost of holding the event for this number of people at either hotel.

(c) The equations are

$$C = 350 + 55N,$$
$$C = 500 + 50N.$$

The easiest way to solve them is to subtract the first from the second. This gives

$$0 = 150 - 5N$$
$$5N = 150$$
$$N = 30.$$

Substituting $N = 30$ into the first equation (the Sandmartin's) gives

$$C = 350 + 55 \times 30 = 350 + 1650 = 2000.$$

(Check: Substituting $N = 30$ into the second equation (the Swift's) gives

$$C = 500 + 50 \times 30 = 500 + 1500 = 2000.)$$

So, as expected, for 30 people the cost is the same for the two hotels; it is £2000 in each case.

This agrees with the answer found in part (b).

(d) The boss would probably like to see the graph; that way, when you give her the information she wants, she will be able to check your answer against the graph, and it will be a useful thing to flash around at meetings!

You should also give her a clear statement explaining how she can make her decision, such as the following:

> If there are 30 participants, then each hotel will charge £2000. If there are fewer than 30 participants, then the Sandmartin should be booked; if more, then the Swift should be booked.

Activity 33

(a) The equations are

$$Q = \tfrac{1}{4}P - 50, \qquad (61)$$
$$Q = -\tfrac{1}{8}P + 70. \qquad (62)$$

Multiplying equation (61) by 4 and equation (62) by 8 gives

$$4Q = P - 200, \qquad (63)$$
$$8Q = -P + 560. \qquad (64)$$

(b) Adding the two equations gives

$$12Q = 360$$
$$Q = 30.$$

Substituting this value of Q into equation (63) gives

$$4 \times 30 = P - 200$$
$$120 = P - 200$$
$$320 = P.$$

So the solution is

$$P = 320, \ Q = 30.$$

(c) The equilibrium price of the apples is £3.20 per kg.

The value of Q obtained in part (b) is the quantity of apples supplied and demanded at the equilibrium price, in millions of kilograms.

Activity 34

If you drive at the maximum speed of 70 mph during the motorway part of the journey, then the time for that part is

$$\frac{\text{distance}}{\text{speed}} = \frac{105}{70} = 1\tfrac{1}{2} \text{ hours.}$$

So you will take at least $1\frac{1}{2}$ hours on the motorway. With half an hour to get to the junction, and a quarter of an hour at the other end, you need to leave at least $2\frac{1}{4}$ hours in total. That is, $t \geq 2\frac{1}{4}$, or $t \geq \frac{9}{4}$.

Activity 35

You certainly can't order as many as 15 CDs, as that would cost over £90 even without considering postage. So your postage will definitely cost £4. So if you order n CDs, you must pay $6.5n + 4$ pounds. Thus an inequality concerning the number of CDs that you can order is

$$6.5n + 4 \leq 75.$$

Activity 36

(a) (i) Yes. Adding 3 to both sides of the inequality $-2 < 1$ gives $1 < 4$, which is correct.

(ii) Yes. Subtracting 4 from both sides of the inequality $-2 < 1$ gives $-6 < -3$, which is correct.

(b) (i) Yes. Multiplying both sides of the inequality $-3 \geq -5$ by 3 gives $-9 \geq -15$, which is correct.

(ii) *No!* Multiplying both sides of the inequality $-3 \geq -5$ by -2 gives $6 \geq 10$, which is wrong.

(c) (i) Yes. Multiplying both sides of the inequality $2 \leq 3$ by 0 gives $0 \leq 0$, which is correct.

(ii) *No!* Multiplying both sides of the inequality $2 < 3$ by 0 gives $0 < 0$, which is wrong.

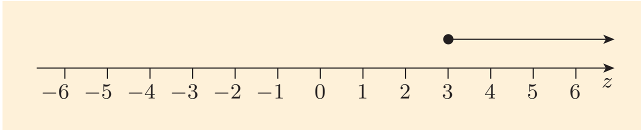
Activity 37

(a) The inequality is: $2z - 1 \geq 5$

Add 1: $2z \geq 6$

Divide by 2: $z \geq 3$

The numbers that satisfy the original inequality can be illustrated on a number line as follows.

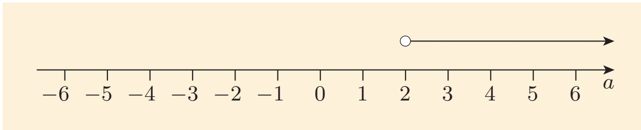


(b) The inequality is: $7 - 3a < 1$

Subtract 7: $-3a < -6$

Divide by -3: $a > 2$

The numbers that satisfy the original inequality can be illustrated as follows.

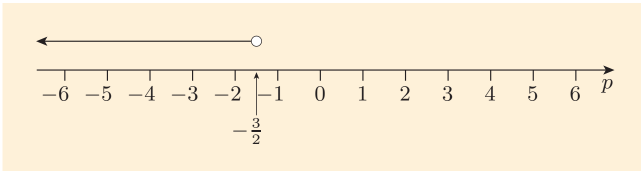


(c) The inequality is: $3 < -2p$

Divide by -2: $-\frac{3}{2} > p$

Swap the sides: $p < -\frac{3}{2}$

The numbers that satisfy the original inequality can be illustrated as follows.



(d) The inequality is: $2 - 3t > t + 1$

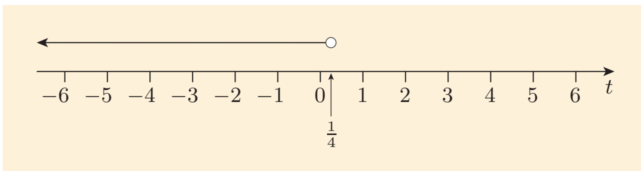
Add $3t$: $2 > 4t + 1$

Subtract 1: $1 > 4t$

Divide by 4: $\frac{1}{4} > t$

Swap the sides: $t < \frac{1}{4}$

The numbers that satisfy the original inequality can be illustrated as follows.



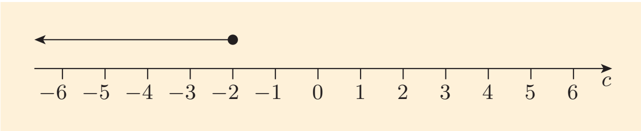
(e) The inequality is: $\frac{1}{2}c - 1 \geq c$

Multiply by 2: $c - 2 \geq 2c$

Subtract c: $-2 \geq c$

Swap the sides: $c \leq -2$

The numbers that satisfy the original inequality can be illustrated as follows.



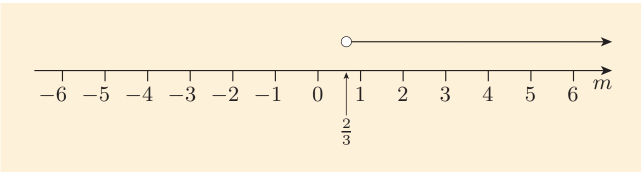
(f) The inequality is: $m > 2(1 - m)$

Multiply out: $m > 2 - 2m$

Add $2m$: $3m > 2$

Divide by 3: $m > \frac{2}{3}$

The numbers that satisfy the original inequality can be illustrated as follows.



Activity 38

The inequality is: $6.5n + 4 \leq 75$

Subtract 4: $6.5n \leq 71$

Divide by 6.5: $n \leq \dfrac{71}{6.5}$

Simplify: $n \leq \dfrac{142}{13}$

Since

$$\frac{142}{13} \approx 10.9,$$

the maximum number of CDs that can be bought is 10.

Activity 39

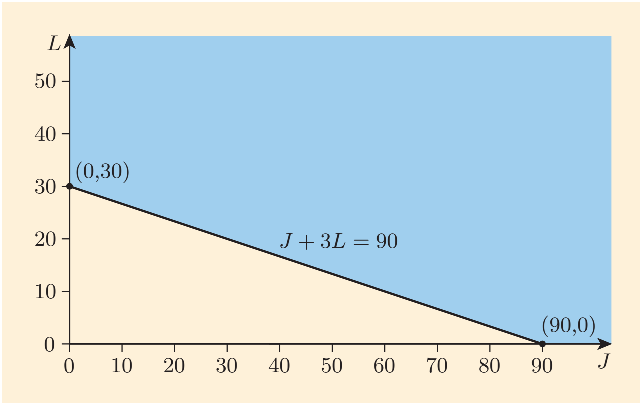
(a) The total quantity of soft drink, in litres, will be

$$J \times 1 + L \times 3 = J + 3L,$$

and you need at least 90 litres, so the inequality is

$$J + 3L \geq 90.$$

(b) The area of practical choice is shown below. It is infinite! This is because you could buy any large numbers of cartons of juice and bottles of lemonade, and the total quantity of drink would always be at least 90 litres.



Activity 40

The point $(10, 30)$ lies within the blue triangle in Figure 15, so one possibility is to buy 10 cartons of juice and 30 bottles of lemonade.

(Check: The cost of these numbers of cartons of juice and bottles of lemonade is

$$£(10 \times 1 + 30 \times 1.50) = £55,$$

which is less than £60, and the quantity of drink that they provide is

$$(10 \times 1 + 30 \times 3) \text{ litres} = 100 \text{ litres},$$

which is more than 90 litres, so both wishes are satisfied.)

Of course, your choice of point could have been slightly different!